Journal of Advances in Mathematics and Computer Science

Volume 38, Issue 10, Page 113-120, 2023; Article no.JAMCS.107049
ISSN: 2456-9968
(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

On Unitary Quasi-equivalence and Partial Isometry Operators in Hilbert Spaces

Anyembe Lilian [a], Musundi Sammy Wabomba [a*] and Kinyanjui Jeremiah Ndung'u [b*]

[a] Department of Physical Sciences, Chuka University, P.O Box-109-60400, Chuka, Kenya.
[b] Department of Pure & Applied Sciences, Kirinyaga University, P.O Box-143-10300, Kerugoya, Kenya.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2023/v38i101829

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here:
https://www.sdiarticle5.com/review-history/107049

Original Research Article

Received: 20/07/2023
Accepted: 26/09/2023
Published: 18/10/2023

Abstract

Unitary quasi-equivalence has been shown to be an equivalence relation. Similarly, unitary quasi-equivalence has been proven to preserve normality, hyponormality and binormality of operators. However, the properties of unitary quasi-equivalence and partial isometric operators have not been established. In this paper therefore, the study aims to determine the properties of unitary quasi-equivalence and isometry, co-isometry and partial isometry operators.

Keywords: Unitary quasi-equivalence; isometry; co-isometry; partial isometry; operator and Hilbert space.

1 Introduction

Let H be a complex Hilbert space and $B(H)$ be a set of bounded linear operators on Hilbert space H. $A \overset{u.q.e}{\approx} B$ and $A \overset{a.s}{\approx} B$ denotes unitary quasi-equivalent and almost similar operators respectively. An operator $V \in B(H)$ is

---

Corresponding author: Email: swmusundi@chuka.ac.ke, jkinyanjui@kyu.ac.ke;

defined as a structure preserving map. For $V \in B(H)$, V^* is called the adjoint of an operator V. Two operators $A, B \in B(H)$ are said to be unitary quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that the following conditions are satisfied:

$$A^*A = UB^*BU^*$$

$$AA^* = UBB^*U^*$$

Nzimbi et al. [1] investigated unitary quasi-equivalence under the concept of near equivalence. This class of equivalence relation operators was later studied further by Ko et al. [2]. It has been established unitary quasi-equivalence to be an equivalence relation operator by Luketero et al. [3]. Luketero et al. [3], also established that unitary equivalence implies unitary quasi-equivalence. However, unless the operators are similar normal, the converse is not always true. Thereafter [4] determined that unitary quasi-equivalence preserve normality of operators. [4] further established that projection unitary quasi-equivalent operators imply that the operators are unitary equivalent. On other classes of equivalence relation such as unitary equivalence and almost similarity it has been established that they preserve partial isometric properties as established [5-8]. Nzimbi et al. [4] established that unitary quasi-equivalence preserves binormality and hyponormality of operators. Nzimbi et al. [4], also established that unitary quasi-equivalence preserves unitary operator properties. That is, if two operators are unitarily quasi-equivalent and one of them is a unitary operator, the other must also be a unitary operator. However, the same results have not been established for isometry, co-isometry and partial isometry operators. As a result, the purpose of this research is to determine the properties of unitary quasi-equivalence on isometry, co-isometry, and partial isometry operators.

2 Definitions and Terminologies

Definition 2.1: A Hilbert space H [9]. A Hilbert space H is a complete inner product space.

Definition 2.2: (Halmos, 2017) [10]. An operator $T \in B(H)$ is said to be

(i) Unitary if $TT^* = T^*T = I$
(ii) Isometry if $T^*T = I$.
(iii) Co-isometry if $TT^* = I$.
(iv) Partial isometry if T^*T is a projection or $T=TT^*T$.

Definition 2.3: [5]. An operator $V \in B(H)$ is said to be partial isometry if

$$V^2 = V^2 (V^2)^* V^2$$

From the above definitions, we have the following class inclusions:

(i) Unitary operators ⊆ isometry operators ⊆ partial isometry operators.
(ii) Unitary operators ⊆ co-isometry operators ⊆ partial isometry operators

Definition 2.4: Unitary quasi-equivalence operators [8]. Two operators $A, B \in B(H)$ are said to be unitarily quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that:

$$A^*A = UB^*BU^*$$

$$AA^* = UBB^*U^*$$

3 Methodology

In achieving this objective, properties of partial isometry operators are useful. Definition of isometry, co-isometry and partial isometry played a vital role. In addition, properties of unitary equivalence on partial isometry operators are also used in comparison. This study aimed to extend the following Lemmas to unitary quasi-equivalence.

Lemma 3.1: [3]. Let $A \in B(H)$ be a partial isometry and $B \in B(H)$ be any other operator such that, either $A = UBU^*$or $A = U^*BU$ where U is unitary, then $B \in B(H)$ is also a partial isometry.

Lemma 3.2: [6]. Let $P, Q \in B(H)$ such that $P \overset{a.s}{\approx} Q$. If P^2 is a partial isometry and Q is self-adjoint, then Q^2 is also partially isometric.

Lemma 3.3: [11]. For an operator $T \in B(H)$, the following statements are equivalent;

(i) T is partial isometry.
(ii) T^* is partial isometry.
(iii) TT^* is projection.
(iv) T^*T is projection.
(v) $T^*TT^* = T.^*$
(vi) $TT^*T = T$.

4 Main Results

The following are results that were established.

Theorem 3.1: If $A, B \in B(H)$ are unitarily quasi-equivalent then A is isometry if and only if B is isometry.

Proof:

Suppose that an operator A is isometry

Since $A \overset{u.q.e}{\approx} B$, by definition [8], these operators satisfy the following conditions.

$$A^*A = UB^*BU^* \dots (i)$$

$$AA^* = UBB^*U^* \dots (ii)$$

But A is isometry thus by definition 2,2 [10].

$$A^*A = I$$

Substituting equation (i) in $A^*A = I$

$$I = A^*A$$

$$= UB^*BU^*$$

Thus

$$UB^*BU^* = I \dots (iii)$$

Pre-multiplying both sides of equation (iii) with U^* and post multiplying with U, it implies

$$U^*UB^*BU^*U = U^*IU$$

$$U^*UB^*BU^*U = U^*U$$

Since an operator U is unitary, then by definition 2.2, [10]

Then, $U^*U = UU^*$

$$= I$$

It implies

$$IB^*BI = I$$

$$B^*B = I$$

Thus B is isometry.

Conversely, suppose that an operator B is isometry, and then by definition 2.2,

$$B^*B = I.$$

However, $A \overset{u.q.e}{\approx} B$ implies that equation (i) and (ii) holds.

Thus substituting $B^*B = I$ in equation (i),

$$A^*A = UB^*BU^*$$

$$= UIU^*$$

$$A^*A = UU^*$$

Since an operator U is unitary then

$$U^*U = UU^*$$

$$= I$$

Implies that

$$UU^* = I.$$

Thus

$$A^*A = I$$

Thus, by definition, an operator A is isometry.

Theorem 3.2: Let $V, W \in B(H)$ be unitarily quasi-equivalent operators, then an operator V is co-isometry if and only if W is co-isometry.

Proof:

Since $V \overset{u.q.e}{\approx} W$, then by definition, [2]

$$V^*V = UW^*WU^* \dots (i)$$

$$VV^* = UWW^*U^* \dots (ii)$$

Suppose V is co-isometry then by definition 2.2, [10]

$$< V^*x, V^*y >=< x, y >, \forall\, x, y \in H.$$

$$VV^* = I$$

Substituting $VV^* = I$ in equation (ii)

$I = VV^*$

$= UWW^*U^*$

$UWW^*U^* = I \dots (iii)$

Pre-multiplying both sides of equation (iii) with U^* and post multiplying with U,

$U^*UWW^*U^*U = U^*IU$

$U^*UWW^*U^*U = U^*U$

But $U^*U = UU^* = I$, since an operator U is unitary

$IWW^*I = I$

$WW^* = I.$

Thus W is co-isometry.

Conversely suppose that an operator W is co-isometry,

Then by definition 1.2,

$WW^* = I$

Substituting $WW^* = I$ in (ii) implies that

$VV^* = UWW^*U^*$

$= UIU^*$

$VV^* = UIU^*$

$VV^* = UU^*$

$VV^* = I$. (Since $UU^* = I$)

Thus V is co-isometry.

Remark: Results one and two implies that unitary quasi-equivalence preserves isometric and co-isometric properties of operators.

Theorem 3.3: If $K, L \in B(H)$ are unitarily quasi-equivalent and K is partial isometry, then an operator L is also a partial isometry operator.

Proof:

Since $K \overset{u.q.e}{\approx} L$, then by definition 1.3, [8],

$K^*K = UL^*LU^* \dots (i)$

$KK^* = ULL^*U^* \dots (ii)$

Since K is partial isometry, then by definition 2.2, [11]. K^*K is a projection**.**

$(K^*K)^2 = K^*K \dots (iii)$

But by equation (i)

$$K^*K = UL^*LU^*$$

Replacing K^*K with UL^*LU^* in equation (iii)

$$UL^*LU^* = (UL^*LU^*)^2$$

$$UL^*LU^* = UL^*LU^*.UL^*LU^*$$

But $U^*U = I$

$$UL^*LU^* = UL^*LIL^*LU^*$$

$$UL^*LU^* = UL^*LL^*LU^*$$

But $L^*LL^*L = (L^*L)^2$

$$UL^*LU^* = U(L^*L)^2U^* \dots (iv)$$

Pre multiplying both sides of equation (iv) with U^* and post multiplying with U, we get,

$$U^*UL^*LU^*U = U^*U(L^*L)^2U^*U$$

But $U^*U = I$

$$IL^*LI = I(L^*L)^2I$$

$$L^*L = (L^*L)^2$$

This implies that L^*L is a projection thus by definition of partial isometry [11] it means that L is partial isometry [12-14].

Theorem 3.4: let $V, W \in B(H)$ be self-adjoint and unitary quasi-equivalence operators, if V^2 is partial isometry so is W^2 partial isometry.

Proof:

Since $V \overset{u.q.e}{\approx} W$, the following conditions hold,

$$V^*V = UW^*WU^* \dots (i)$$

$$VV^* = UWW^*U^* \dots (ii)$$

But V, W are self adjoint unitary quasi-equivalent operators, [4] established that V^2, W^2 are unitarily equivalent. That is by (i) and (ii) and definition of self-adjoint operator [15-17],

$$\begin{aligned} V^*V &= VV^* \\ &= VV \\ &= V^2 \end{aligned}$$

and

$$\begin{aligned} W^*W &= WW^* \\ &= WW \\ &= W^2 \end{aligned}$$

Implying that,

$$V^2 = UW^2U^*$$

Since V^2 is partial isometry then by definition of partial isometry [5]

$$V^2 = V^2{V^2}^*V^2 \ldots (iii)$$

But $V^2 = UW^2U^*$

Replacing V^2 with UW^2U^* in equation (iii)

$$UW^2U^* = UW^2U^*(UW^2U^*)^*UW^2U^*$$

But $(UW^2U^*)^* = U{W^2}^*U^*$

$$UW^2U^* = UW^2U^*U{W^2}^*U^*UW^2U^*$$

$UW^2U^* = UW^2I{W^2}^*IW^2U^*$, (since $U^*U = I$)

Thus

$$UW^2U^* = UW^2{W^2}^*W^2U^* \ldots (iv)$$

Pre-multiplying both sides of equation (iv) by U^* and post multiplying by U,

We have

$$U^*UW^2U^*U = U^*UW^2{W^2}^*W^2U^*U$$

But $U^*U = I$

$$IW^2I = IW^2{W^2}^*W^2I$$

$$W^2 = W^2{W^2}^*W^2 \ldots (v)$$

Equation (v) implies that W^2 is a partial isometric operator.

5 Conclusion

Based on the preceding results, it establishes that unitary quasi-equivalent operators preserve; isometry, co-isometry and partial isometric properties. That is, if two operators are unitarily quasi-equivalent and one of them is isometry, co-isometry, or partial isometry, then the other operator is also isometry, co-isometry, or partial isometry.

Conference Disclaimer

Some part of this manuscript was previously presented and published in the conference: 9th International Research Conference dated 24th and 25th November, 2022 Kenya, Web Link of the proceeding: https://www.chuka.ac.ke/storage/2023/05/9th-IRC-Proceedings2022-2023.pdf

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Nzimbi BM, Wanyonyi SL. On unitary quasi-equivalence of operators. International Journal of Mathematics And its Applications. 2020;8(1):207-215.

[2] Ko E, Lee MJ. Remarks on nearly equivalent operators. Operators and Matrices. 2018;12(2):537-547.

[3] Luketero SW, Khalagai JM. On unitary equivalence of some classes of operators in Hilbert spaces; 2020.

[4] Nzimbi BM, Pokhariyal GP, Khalaghai JM. A note on similarity, almost-similarity and equivalence of operators. FJMS. 2008;28(2):305-319,11.

[5] KD D, Pradhan DK, Sarkar J. Partially isometric toeplitz operators on the polydisc. Bulletin of the London Mathematical Society. 2022;54(4):1350-1362.

[6] Kiprop SK, Njue King'ang'I D, Mutekhele J. On almost similarity results on partial isometrics, operators and posinormal operators; 2021.

[7] Salhi A, Zerouali E. Decomposition of partial isometries with finite ascent. Advances in Operator Theory. 2020;5:64-82.

[8] Kutkut M. On quasi-equivalent operators. Bull. Cal. Math. Soc. 1998;90:45-52.

[9] Berberian SK. Introduction to Hilbert space (Vol. 287). American Mathematical Soc; 1999.

[10] Halmos PR. Introduction to Hilbert space and the theory of spectral multiplicity. Courier Dover Publications; 2017.

[11] Okelo BN. The norm attainability of some elementary operators. Applied Mathematics E-Notes; 2012.

[12] Faris WG. Self-adjoint operators. Springer. 2006;433.

[13] Kinyanjui JN, Okelo NB, Ongati O, Musundi SW. Norm Estimates for Norm-Attainable Elementary Operators. International Journal of Mathematical Analysis. 2018;12(3):137-144.

[14] Musundi SW, Sitati N, Nzimbi BM, Murwayi AL. On almost similarity operator equivalence relation; 2013.

[15] Muteti IM. On equivalence of some operators in Hilbert spaces (master's thesis, School of Mathematics, University of Nairobi); 2014.

[16] Nzimbi BM, Luketero SW, Sitati IN, Musundi SW, Mwenda E. On almost similarity and metric equivalence of operators. be published by Pioneer Journal of Mathematics and Mathematical sciences (June 14, 2016); 2016.

[17] Skoufranis P. Numerical ranges of operators. Lecture Notes. Department of Mathematics and Statistics, York University. 2014;34:93-94.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/107049