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Normalized Solutions of Mass-Subcritical Schrödinger-Maxwell Equations

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Abstract

In this paper, we investigate the existence of normalized solutions to the coupling of the nonlinear Schrödinger-Maxwell equations. In the mass-subcritical case, we by weak lower semmicontinuity of norm prove that the equations satisfying normalization condition exist a normalized ground state solution.

Subject Areas

Mathematics

Keywords

Normalized Solutions, Schrödinger-Maxwell Equations

1. Introduction

In this paper, we study the existence of normalized ground state solution of the following Schrödinger-Maxwell equations

$$\begin{cases} -\Delta u + u + \phi u + \lambda u = f(u) \text{ in } \mathbb{R}^N, \\ -\Delta \phi = u^2 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\phi : \mathbb{R}^N \to \mathbb{R}$ and $2 < N < 6$, the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. The unknowns of the equations are the field u associated to the particle and the electric potential ϕ, and satisfying the normalization condition

$$\int_{\mathbb{R}^N} |u|^2 \, dx = a, \tag{1.2}$$

we prescribe $a > 0$. Hence, we have

$$\begin{cases} -\Delta u + u + \phi u + \lambda u = f(u) \text{ in } \mathbb{R}^N, \\ -\Delta \phi = u^2 \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, dx = a. \end{cases} \tag{1.3}$$

where u belongs to the Hilbert space

$$\mathcal{H} = \left\{ u \in H_r^1\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \mathrm{d}x < \infty \right\},$$

and

$$H_r^1\left(\mathbb{R}^N\right) = \left\{ u \in H^1\left(\mathbb{R}^N\right) : u(x) = u\left(|x|\right) \right\}.$$

The space $\mathcal{H}$ is endowed with the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right) \mathrm{d}x.$$

Let $D^{1,2} \equiv D^{1,2}\left(\mathbb{R}^N\right) = \left\{ u \in L^{2^*}\left(\mathbb{R}^N\right) : \nabla u \in L^2\left(\mathbb{R}^N\right) \right\}$ with respect to the norm

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x.$$

For any $2 < s < 2^*$, $L^s\left(\mathbb{R}^N\right)$ is endowed with the norm

$$|u|_s^s = \int_{\mathbb{R}^N} |u|^s \mathrm{d}x.$$

Obviously, the embedding $\mathcal{H} \hookrightarrow L^s\left(\mathbb{R}^N\right)$ is compact (see [1]).

By the variational nature, the weak solutions of (1.1) are critical points of the functional $J : \mathcal{H} \times D^{1,2} \to \mathbb{R}$ defined by

$$J(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 \right) \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^N} |\nabla \phi|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \phi u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \mathrm{d}x,$$

where $F(t) = \int_0^t f(s)\mathrm{d}s$ is a rather general nonlinearity. Then, it is clear that the function J is C^1 on $\mathcal{H} \times D^{1,2}$ and has the strong indefiniteness. We can know that the weak solutions of (1.1) $(u,\phi) \in \mathcal{H} \times D^{1,2}$ are critical points of the functional J. By standard arguments, the function J is C^1 on $\mathcal{H} \times D^{1,2}$.

In recent years, normalized solutions of Schrödinger equations have been widely studied. When searching for the existence of normalized solutions of Schrödinger equations in $\mathbb{R}^N$, appears a new mass-critical exponent

$$l = 2 + \frac{4}{N}.$$

Now, let us review the involved works. In the mass-subcritical case, Zuo Yang and Shijie Qi [2] proved that for all $a > 0$, the following Schrödinger equations with potentials and non-autonomous nonlinearities

$$\begin{cases} -\Delta u + V(x)u + \lambda u = f(x,u) \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, u \in H^1\left(\mathbb{R}^N\right), \end{cases}$$

have a normalized solutions. Nicola Soave [3] in the mass-subcritical proved the nonlinear Schrödinger equation with combined power nonlinearities mass- critical and mass-supercritical cases studied of:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{p-2} u + |u|^{2^*-2}, u \text{ in } \mathbb{R}^N, N \ge 3, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, u \in H^1\left(\mathbb{R}^N\right). \end{cases}$$

have several stability/instability and existence/non-existence results of normalized ground state solutions. For $g(u)$ is a superlinear, subcritical, Thomas Bartsch [4] studied the existence of infinitely many normalized solutions for the problem

$$-\Delta u - g(u) = \lambda u, u \in H^1(\mathbb{R}^N),$$

By establishing the compactness of the minimizing sequences, Tianxiang Gou and Louis Jeanjean [5] in the mass-subcritical studied the existence of multiple positive solutions to the nonlinear Schrödinger systems:

$$\begin{cases} -\Delta u = \lambda_1 u + \mu_1 |u_1|^{p_1-2} u_1 + \beta r_1 |u|^{r_1-2} u_1 |u_2|^{r_2}, \\ -\Delta u = \lambda_2 u + \mu_2 |u_2|^{p_2-2} u_2 + \beta r_2 |u|^{r_1} |u_2|^{r_2-2} u_2. \end{cases}$$

In the mass-subcritical case, Masataka Shibata [6] studied for the nonlinear Schrödinger equations with the minimizing problem:

$$E_a = \inf\left\{ I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(|u|)\mathrm{d}x \mid u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a \right\}$$

where $F(t) = \int_0^t f(s)\mathrm{d}s$ is a general nonlinear term. They proved E_a is attained. That is to say, the Schrödinger equations have normalized solutions. Moreover, for the $I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(|u|)\mathrm{d}x$ case, Norihisa Ikoma and Yasuhito Miyamoto [7] showed the existence of the minimizer of the minimization problem E_a, where $V(x) \to 0$ as $|x| \to \infty$. They also obtained the conclusions that the normalized solutions of Schrödinger equations exist. In the mass-subcritical condition, Zhen Chen and Wenming Zou [8] basing on the refined energy estimates proved the existence of normalized solutions to the Schrödinger equations.

Other related normalized solutions problems of Schrödinger can be seen in [9] [10] [11] [12] [13]. Thus, the main purpose of this paper is to study the solution of Schrödinger-Maxwell equations satisfying normalization condition by using above results. In particular, the situation we consider will involve the presence of potential ϕ. In addition, the nonlinear term $f(u)$ is mass-subcritical and satisfies the following appropriate assumptions. In this case, the functional I is bounded from below and coercive on $S(a)$, which will be proved in Lemma 2.5.

We assume the following conditions throughout the paper:

($f1$) $f: \mathbb{R}^N \to \mathbb{R}$ is continuous.

($f2$) $\lim_{s\to 0} \frac{f(s)}{s} = 0$ and $\lim_{|s|\to+\infty} \frac{f(s)}{|s|^{l-1}} = 0$ with $l = 2 + \frac{4}{N}$.

Moreover, c and c_i are positive constants which may change from line to line.

Our main result is the following theorem:

Theorem 1.1 *Suppose* ($f1$) *and* ($f2$) *hold. Then, for any* $a > 0$, *problem* (1.3) *has a normalized ground state solution.*

2. Proof of Main Results

Since the functional J exhibits a strong indefiniteness. To avoid the difficulty we use the reduction method. Thus, we shall introduce the method.

For any $u \in \mathcal{H}$, us consider the linear operator $T(u): D^{1,2} \to \mathbb{R}$ defined as

$$T(u) = \int_{\mathbb{R}^N} u^2 v \mathrm{d}x. \tag{2.1}$$

Then, there exists a positive constant c_1 such that

$$\int_{\mathbb{R}^N} u^2 v \mathrm{d}x \le \|u^2\|_{L^{\frac{2N}{N+2}}} \|v\|_{L^{2^*}} \le \|u\|^2_{L^{\frac{4N}{N+2}}} \|v\|_{L^{2^*}} \le c_1 \|u\|^2_{\mathcal{H}} \|v\|_{D^{1,2}},$$

because the following embeddings are continuous:

$$\mathcal{H} \hookrightarrow L^s(\mathbb{R}^N), \quad \forall s \in [2, 2^*] \quad \text{and} \quad D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N).$$

We set

$$g(\varphi, v) = \int_{\mathbb{R}^N} \nabla \varphi \cdot \nabla v \mathrm{d}x, \varphi, v \in D^{1,2}.$$

Obviously, $g(\varphi, v)$ is linear in φ and v respectively.

Moreover, there exists a positive constant c_2 and c_3 such that for any $\varphi, v \in D^{1,2}$,

$$|g(\varphi, v)| \le c_2 \|\varphi\|_{D^{1,2}} \|v\|_{D^{1,2}}, \tag{2.2}$$

$$g(\varphi, v) \ge c_3 \|\varphi\|^2_{D^{1,2}}. \tag{2.3}$$

Combining (2.2) and (2.3) we know that $g(\varphi, v)$ is bounded and coercive. Hence, by the Lax-Milgram theorem we have that for every $u \in \mathcal{H}$, for any $v \in D^{1,2}$, there exists a unique $\phi_u \in D^{1,2}$ such that

$$T(u)v = g(\phi_u, v).$$

Then, for any $v \in D^{1,2}$, we obtain

$$\int_{\mathbb{R}^N} u^2 v \mathrm{d}x = \int_{\mathbb{R}^N} \nabla \phi_u \cdot \nabla v \mathrm{d}x, \tag{2.4}$$

and using integration by parts, we have

$$\int_{\mathbb{R}^N} \nabla \phi_u \cdot \nabla v \mathrm{d}x = -\int_{\mathbb{R}^N} v \Delta \phi_u \mathrm{d}x.$$

Therefore,

$$-\Delta \phi_u = u^2 \tag{2.5}$$

in a weak sense, and ϕ_u has the following integral expression:

$$\phi_u = \frac{1}{4\pi} \int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|} \mathrm{d}y, \tag{2.6}$$

The functions ϕ_u possess the following properties:

Lemma 2.1 *For any* $u \in \mathcal{H}$, *we have:*

1) $\|\phi_u\|_{D^{1,2}} \le c_4 \|u\|^2_{L^{\frac{4N}{N+2}}}$, *where* $c_4 > 0$ *is independent of u. As a consequence there exists* $c_5 > 0$ *such that*

$$\int_{\mathbb{R}^N} \phi_u u^2 \mathrm{d}x \le c_5 \|u\|^4_{\mathcal{H}};$$

2) $\phi_u \ge 0$.

Proof. 1) For any $u \in \mathcal{H}$, using (2.5) we have

$$\|\phi_u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla \phi_u|^2 \mathrm{d}x = -\int_{\mathbb{R}^N} \phi_u \Delta \phi_u \mathrm{d}x = \int_{\mathbb{R}^N} \phi_u u^2 \mathrm{d}x$$
$$\le \|\phi_u\|_{L^{2^*}} \|u^2\|_{L^{\frac{2N}{N+2}}} \le c_4 \|\phi_u\|_{D^{1,2}} \|u\|_{L^{\frac{4N}{N+2}}}^2,$$

where c_4 is a positive constant. Hence, we obtain that

$$\|\phi_u\|_{D^{1,2}} \le c_4 \|u\|_{L^{\frac{4N}{N+2}}}^2,$$

therefore there exists a positive constant c_5 such that

$$\int_{\mathbb{R}^N} \phi_u u^2 \mathrm{d}x \le c_4 \|\phi_u\|_{D^{1,2}} \|u\|_{L^{\frac{4N}{N+2}}}^2 \le c_4^2 \|u\|_{L^{\frac{4N}{N+2}}}^4 \le c_5 \|u\|_{\mathcal{H}}^4, \tag{2.7}$$

because we know for any $s \in [2, 2^*]$, $\mathcal{H} \hookrightarrow L^s(\mathbb{R}^N)$.

2) Obviously, by the expression (2.6) the conclusion holds. □

Now let us consider the functional $I : \mathcal{H} \to \mathbb{R}^N$,

$$I(u) := J(u, \phi_u).$$

Then I is C^1.

By the definition of *J*, we have

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^N} |\nabla \phi_u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

Multiplying both members of (2.5) by ϕ_u and integrating by parts, we obtain

$$\int_{\mathbb{R}^N} |\nabla \phi_u|^2 \mathrm{d}x = \int_{\mathbb{R}^N} \phi_u u^2 \mathrm{d}x.$$

Therefore, the functional I may be written as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x. \tag{2.8}$$

The following lemma is Proposition 2.3 in [5].

Lemma 2.2 *The following statements are equivalent*:

1) $(u, \phi) \in \mathcal{H} \times D^{1,2}(\mathbb{R}^N)$ *is a critical point of J*.

2) u *is a critical point of I and* $\phi = \phi_u$.

Hence u is a solution to (1.3) if and only if u is the critical point of the functional (2.8). The critical point can be obtained as the minimizer under the constraint of L^2-*sphere*

$$S(a) = \left\{u \in \mathcal{H} : \int_{\mathbb{R}^N} u^2 \mathrm{d}x = a\right\}. \tag{2.9}$$

We shall study the constraint problem as follows:

$$E_a = \inf_{u \in S(a)} I(u). \tag{2.10}$$

The solution of (13) $u = \tilde{u}$ is called a normalized ground state solution satisfying problem (3) if it has minimal energy among all solutions:

$$dI|_{S(a)}(\tilde{u}) = 0 \text{ and } I(\tilde{u}) = \inf\left\{I(u) : dI|_{S(a)}(\tilde{u}) = 0, \tilde{u} \in S(a)\right\}.$$

In this paper, we will be especially interested in the existence of normalized ground state solutions.

Lemma 2.3 *We define* $\Phi : \mathcal{H} \to D_r^{1,2}$, $\Phi(u) = \phi_u$, *which is also the solution* *of the Equation* (2.5) *in* $D^{1,2}$. *Let* $\{u_n\}\subset S(a)$ *be a minimizing sequence of I with satisfying* $u_n \rightharpoonup u$ *in* $\mathcal{H}$. *Then,* $\Phi(u_n)\to\Phi(u)$ *in* $D^{1,2}$ *and we obtain*

$$\int_{\mathbb{R}^N}\Phi(u_n)u_n^2\mathrm{d}x\to\int_{\mathbb{R}^N}\Phi(u)u^2\mathrm{d}x \quad as \quad n\to\infty. \tag{2.11}$$

Proof. By (2.1), the following expressions hold

$$T(u_n)v=\int_{\mathbb{R}^N}u_n^2v\mathrm{d}x, T(u)v=\int_{\mathbb{R}^N}u^2v\mathrm{d}x.$$

Since $u\in\mathcal{H}$ and the embedding $H_r^1\hookrightarrow L^s$ is compact for any $s\in(2,2^*)$, clearly we have

$$u^2\in L^1(\mathbb{R}^N)\cap L^N(\mathbb{R}^N), \tag{2.12}$$

then, by interpolation we have

$$u^2\in L^{\frac{N}{2}}(\mathbb{R}^N).$$

Using again (2.12), we get

$$u^2\in L^{\frac{2N}{N+2}}(\mathbb{R}^N).$$

Moreover, $\{u_n\}$ be a minimizing sequence and $u_n\rightharpoonup u$ in $\mathcal{H}$, we obtain

$$u_n^2\to u^2 \text{ in } L^{\frac{2N}{N+2}}. \tag{2.13}$$

Therefore, we get

$$\left|T(u_n)v-T(u)v\right|=\left|\int_{\mathbb{R}^N}u_n^2v\mathrm{d}x-\int_{\mathbb{R}^N}u^2v\mathrm{d}x\right|\le\left|u_n^2-u^2\right|_{L^{\frac{2N}{N+2}}}\left|v^6\right|_{L^{2^*}},$$

which implies that $T(u_n)$ converges strongly to $T(u)$.

Hence, we obtain

$$\Phi(u_n)\to\Phi(u) \text{ in } D^{1,2},$$

$$\Phi(u_n)\to\Phi(u) \text{ in } L^{2^*}. \tag{2.14}$$

By (2.13) and (2.14), we know that conclusion (2.11) holds. □

Lemma 2.4 (*Gagliardo-Nirenberg inequality*). *For all* $u\in\mathcal{H}$, *we have*

$$\|u\|_p^p\le C(N)\|\nabla u\|_{p'}^2\|u\|_2^{p-p'}, 2<p<2^*$$

where $C(N)$ *is a positive constant depending on N and* $p'=\dfrac{N(p-2)}{2p}$.

Lemma 2.5 *Suppose* (*f*1) *and* (*f*2) *hold, than for any* $a>0$, *the functional I is bounded from below and coercive on* $S(a)$.

Proof. Assumptions (*f*1) and (*f*2) imply that for any $\varepsilon>0$, there exist $C_\varepsilon>0$ such that

$$F(s)\le C_\varepsilon|s|^2+\varepsilon|s|^l, \forall s\in\mathbb{R}.$$

Hence, according to Lemma 2.4 with $p=l=2+\dfrac{4}{N}$, we obtain that

$$\left|\int_{\mathbb{R}^N}F(u)\mathrm{d}s\right|\le C_\varepsilon\|u\|_2^2+\varepsilon\|u\|_l^l$$

$$\le C_\varepsilon\|u\|_2^2+\varepsilon C(N)\|\nabla u\|_2^2\|u\|_2^{\frac{4}{N}}$$

Choose ε such that $\varepsilon C(N) a^{\frac{2}{N}} = \frac{1}{4}$, than

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x \\ &\ge \frac{1}{4}\|\nabla u\|_2^2 - Ca > -\infty \end{aligned}$$

Therefore, I is bounded from below and coercive on $S(a)$. □

The following lemma is Lemma 2.2 in [6].

Lemma 2.6 *Suppose* ($f1$) *and* ($f2$) *hold and* $\{u_n\}_{n\in N}$ *is a bounded sequence in* $\mathcal{H}$. *If* $\lim_{n\to\infty} |u_n|_2^2 = 0$ *holds, then it is true that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x = 0.$$

Next, we collect a variant of Lemma 2.2 in [14]. The proof is similar, so we omit it.

Lemma 2.7 *Suppose* ($f1$) *and* ($f2$) *hold and* $\{u_n\}_{n\in N}$ *is a bounded sequence in* $\mathcal{H}$, *then we have* $u_n \rightharpoonup u$ *in* $\mathcal{H}$, *thus*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \left[F(u_n) - F(u) - F(u_n - u)\right]\mathrm{d}x = 0.$$

Proof of Theorem 1.1. Let $\{u_n\} \subset S(a)$ be a minimizing sequence of I with concerning E_a. Then, by (9) we obtain

$$I(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + u_n^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^n} \phi_{u_n} u_n^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x.$$

According to Lemma 2.5, the sequence $\{u_n\}$ is bounded in $\mathcal{H}$. Letting u_0 be in $\mathcal{H}$. Moreover, we know that the embedding $\mathcal{H} \hookrightarrow L^s(\mathbb{R}^N)$ is compact. Hence, we conclude

$$u_n \rightharpoonup u_0 \text{ in } \mathcal{H}, \tag{2.15}$$

$$u_n \to u_0 \text{ in } L^s(\mathbb{R}^N), 2 < s < 2^*, \tag{2.16}$$

$$u_n \to u_0 \text{ a.e. in } \mathbb{R}^N.$$

We also have

$$I(u_0) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_0|^2 + u_0^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^n} \phi_{u_0} u_0^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u_0)\mathrm{d}x.$$

Since (19) holds, we have $\lim_{n\to\infty} |u_n - u_0|_2^2 = 0$. Then, by Lesmma 2.6 we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n - u_0)\mathrm{d}x = 0.$$

Moreover, by Lemma 2.7 we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \left[F(u_n) - F(u_0)\right]\mathrm{d}x = 0.$$

which implies

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^N} F(u_0)\mathrm{d}x \text{ as } n\to\infty. \tag{2.17}$$

Hence, combining weak lower semicontinuity of the norm $\|\cdot\|_{\mathcal{H}}$, Lemma 2.3 and (2.17), we have

$$E_a \le I(u_0) \le \liminf_{n\to\infty} I(u_n) = E_a,$$

which implies $I(u_0) = E_a$. Then, u_0 satisfies

$$\begin{cases} -\Delta u_0 + u_0 + \phi u_0 + \lambda u_0 = f(u_0) \text{ in } \mathbb{R}^N, \\ -\Delta\phi = u_0^2 \text{ in } \mathbb{R}^N, \end{cases}$$

and $\int_{\mathbb{R}^N} |u_0|^2 \,\mathrm{d}x = a$. Therefore, problem (1.3) has a normalized ground state solution. □

Conflicts of Interest

The author declares no conflicts of interest.

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