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Some Fixed Point Theorems for Berinde-Type Contraction Mappings on G_p-Metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we define the concepts of $(\delta, 1-\delta)$-weak contraction, $(\varphi, 1-\delta)$-weak contraction and Ćirić-type almost contraction in the sense of Berinde in G_p-complete G_p-metric space. Furthermore, we prove the existence of fixed points and common fixed points of mappings satisfying Berinde-type contractions stated above and also provide the conditions which are necessary for the uniqueness of the fixed points and common fixed points. Consequently, we obtain the generalizations of comparable results in the literature. In addition, we introduce a few examples which ensure the existence of these attained results.

Keywords: Fixed point; common fixed point; G_p-metric space; berinde-type almost contractions.

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1 Introduction

Fixed-point theory has become one of the fundamental subject of studies which gathers the attention of many scientists within and outside mathematics. It has enormous amount of applications in the fields beside mathematics such as biology, chemistry, physics, economics, computer sciences and engineering which allow rapid advances in a short span of time and improve the existent ideas by providing a wide range of practice possibilities. In 1922, Banach [1] established the fixed point theory and named it as the Banach contraction theory. From then on, a lot of fixed point theorems for different types of contractions came to light. Some of these fixed point theorems were introduced by Berinde. Berinde [2, 3, 4] presented challenging fixed point theorems for different kinds of contraction mappings. Firstly in 2004, Berinde introduced the almost contradiction also known as the weak contraction. Later, in [4], by using comparison function, he defined the concept of φ-almost contraction which is also addressed as (φ, L)-weak contraction. Aside from Berinde, in 1974, Ćirić presented some fixed point theorems by defining Ćirić-type almost contractions which are regarded as one of the most general contractions. Most recently, Ampadu [5] introduced a new type contraction which is called $(\delta, 1-\delta)$-weak contraction.

In addition to the classical concepts of metric space on almost contractions mentioned above, there are some generalizations of metric spaces. One of these generalizations is partial metric space. In 1994, Matthews [6] introduced this concept which differentiated from metric space as it claimed the self-distance is not necessarily zero. Later in 2005, another generalization was introduced by Mustafa and Sims [7] which is known as G-metric space. The latest generalization which constitute a combination of both partial metric space and G-metric space is established by Zand and Nezhad [8]. As the continuation, Aydi et al. [9] familiarized some fixed point results in G_p-metric spaces which is regarded as the source of fixed point results in G_p-metric spaces. Based on the notion of a G_p-metric space, many fixed point results for mappings satisfying various contractive conditions have been presented, for more detailed information (see [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]).

The aim of this study is to construct fixed point and common fixed point theorems for $(\delta, 1-\delta)$-weak contraction, $(\varphi, 1-\delta)$-weak contraction and Ćirić-type almost contraction in the sense of Berinde in G_p-complete G_p-metric space. Moreover, we provide some conditions to attain unique fixed point and common fixed points. The results we obtain extend and generalize some of the results in the literature. Lastly, we present a few examples to illustrate the usability of our obtained results.

2 The Basic Results and Definitions

The aim of this section is to present some preliminary definitions, concepts and theorems used in the paper. First, we provide some basic definitions and properties of G_p-metric space.

Recently, a new generalization and unification of both partial metric space and a G-metric space is introduced by Zand and Nezhad [8]. They named this new space as G_p-metric space and defined it in the following way. We will use the following definition of a G_p-metric space.

Definition 2.1. [8] Let X be a nonempty set. A function $G_p : X \times X \times X \to [0, +\infty)$ is called a G_p-metric space if the following conditions are satisfied:

G_{p_1}. If $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$, then $x = y = z$;

G_{p_2}. $0 \leq G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$;

G_{p_3}. $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) = \dots$, symmetry in all three variables;

G_{p_4}. $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ for any $x, y, z, a \in X$.

Then the pair of (X, G_p) is called a G_p-metric space.

Remark 2.1. With G_{p_2} assumption, it is very easy to show that

$$G_p(x, y, y) = G_p(x, x, y)$$

holds for all $x, y \in X$, i.e., the respective space is symmetric.

An easy example of G_p-metric space is given as follows:

Example 2.1. *[8] Let $X = [0, \infty)$ and define $G_p(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then, (X, G_p) is a symmetric G_p-metric space.*

Some of the properties of G_p-metric space are given in the following proposition.

Proposition 2.1. *[8] Let (X, G_p) be a G_p-metric space, then for any x, y, z and $a \in X$, the followings hold:*

i. $G_p(x, y, z) \leq G_p(x, x, y) + G_p(x, x, z) - G_p(x, x, x)$;

ii. $G_p(x, y, y) \leq 2G_p(x, x, y) - G_p(x, x, x)$;

iii. $G_p(x, y, z) \leq G_p(x, a, a) + G_p(y, a, a) + G_p(z, a, a) - 2G_p(a, a, a)$;

iv. $G_p(x, y, z) \leq G_p(x, a, z) + G_p(a, y, z) - G_p(a, a, a)$.

The following proposition proves that we can link every G_p-metric space to one particular metric space.

Proposition 2.2. *[8] Every G_p-metric space (X, G_p) defines a metric space (X, D_{G_p}),*

$$D_{G_p}(x, y) = G_p(x, y, y) + G_p(y, x, x) - G_p(x, x, x) - G_p(y, y, y)$$

for all $x, y \in X$.

Zand and Nezhad [8] also defined the basic topological concept of G_p-convergence in G_p-metric spaces as the following.

Definition 2.2. Let (X, G_p) be a G_p-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ said to be the limit of the sequence $\{x_n\}$ and denoted by $x_n \to x$ if

$$\lim_{m,n\to\infty} G_p(x, x_n, x_m) = G_p(x, x, x).$$

In this case, we say that the sequence $\{x_n\}$ is G_p-convergent to x.
Thus, if $x_n \to x$ in a G_p-metric space (X, G_p), then for any $\varepsilon > 0$, there exists $l \in \mathbb{N}$ such that

$$|G_p(x, x_n, x_m) - G_p(x, x, x)| < \varepsilon$$

for all $n, m > l$.

By using the above definition, the following proposition can be proved. Moreover, this proposition will play a crucial role in obtaining our results.

Proposition 2.3. *[8] Let (X, G_p) be a G_p-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$ the followings are equivalent:*

i. *$\{x_n\}$ is G_p-convergent to x;*

ii. $G_p(x_n, x_n, x) \to G_p(x, x, x)$ *as* $n \to \infty$;

iii. $G_p(x_n, x, x) \to G_p(x, x, x)$ *as* $n \to \infty$.

Through the definition of D_{G_p}, the following proposition can be deduced.

Proposition 2.4. *Let (X, G_p) be a G_p-metric space. Then, for any sequence $\{x_n\}$ in X G_p-convergent to a point $x \in X$ such that $\lim_{n\to\infty} G_p(x_n, x_n, x_n) = G_p(x, x, x)$ then $D_{G_p}(x_n, x) \to 0$.*

Zand and Nezhad [8] also defined some basic topological concepts like G_p-Cauchy sequence and G_p-completeness in G_p-metric spaces as follows.

Definition 2.3. Let (X, G_p) be a G_p-metric space.

i. A space $\{x_n\}$ is called G_p-Cauchy sequence if and only if $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$ exists (and is finite);

ii. A G_p-metric space (X, G_p) is said to be G_p-complete if and only if every G_p-Cauchy sequence in X is G_p-converges to $x \in X$ such that

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x, x, x).$$

In order to obtain our main results, we need following lemmas.

Lemma 2.2. *[9] Let (X, G_p) be a G_p-metric space.*

i. *If $G_p(x, y, z) = 0$, then $x = y = z$;*

ii. *If $x \neq y$, then $G_p(x, y, z) > 0$.*

Proof. Let $G_p(x, y, z) = 0$. Then, by G_{p_2} we get

$$0 \leq G_p(z, z, z), G_p(y, y, y), G_p(x, x, x) \leq G_p(x, y, z) = 0.$$

Therefore, we get $G_p(x, y, z) = G_p(x, x, x) = G_p(y, y, y) = G_p(z, z, z) = 0$. By G_{p_1}, we conclude that $x = y = z$. Thus, **i** holds.

On the other hand, let $x \neq y$ and $G_p(x, y, z) = 0$. Then, by **i**, $x = y$, which is a contradiction. Hence, **ii** holds. □

Lemma 2.3. *[9] Assume that $\{x_n\} \to x$ as $n \to \infty$ in a G_p-metric space (X, G_p) such that $G_p(x, x, x) = 0$. Then, for every $x, y \in X$*

i. $\lim_{n\to\infty} G_p(x_n, y, y) = G_p(x, y, y)$.

ii. $\lim_{n\to\infty} G_p(x_n, x_n, y) = G_p(x, x, y)$.

The following definition and proposition which were described by Zand and Nezhad will be useful in the process.

Definition 2.4. [8] Let (X_1, G_1) and (X_2, G_2) be two G_p-metric spaces and let $f : (X_1, G_1) \to (X_2, G_2)$ be a function. Then, f is said to be G_p-continuous at a point $a \in X_1$ if and only if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X_1$ and $G_1(a, x, y) < \delta + G_1(a, a, a)$ implies that $G_2(f(a), f(x), f(y)) < \varepsilon + G_2(f(a), f(a), f(a))$. A function f is G_p-continuous on X_1 if and only if it is G_p-continuous at all $a \in X_1$.

Proposition 2.5. *[8] Let (X_1, G_1) and (X_2, G_2) be two G_p-metric spaces. Then, a function $f : X_1 \to X_2$ is G_p-continuous at a point $x \in X_1$ if and only if it is G_p-sequentially continuous at a x; that is, whenever $\{x_n\}$ is G_p-convergent to x one has $\{f(x_n)\}$ is G_p-convergent to $f(x)$.*

The following lemma, which was given by Parvaneh et al. in [10], provides the characterizations of concepts of Cauchy and completeness for G_p-metric spaces.

Lemma 2.4. *[10]*

i. *A sequence $\{x_n\}$ is a G_p-Cauchy sequence in a G_p-metric space (X, G_p) if and only if it is a Cauchy sequence in the metric space (X, D_{G_p});*

ii. *A G_p-metric space (X, G_p) is G_p-complete if and only if the metric space (X, D_{G_p}) is complete. Moreover $\lim_{n\to\infty} D_{G_p}(x, x_n) = 0$ if and only if*

$$\begin{aligned}\lim_{n\to\infty} G_p(x, x_n, x_n) &= \lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_n, x_m)\\ &= \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x, x, x).\end{aligned}$$

The concepts of comparison function and (c)-comparison function which play significant role in forming some of our results are defined as follows.

Definition 2.5. [2] Let $\varphi : [0, \infty) \to [0, \infty)$ be a function. If

i_φ. φ is monotone increasing, that is, $t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$

and

ii_φ. for all $t \geq 0$, $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero,

then φ is called a comparison function. Furthermore, if φ satisfies both i_φ and the following condition

iii_φ. the series $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$,

then φ is called a (c)-comparison function.

From above definition, it is easy to notice that every (c)-comparison function is also a comparison function.

Lemma 2.5. *[2] If φ is a comparison function then $\varphi(t) < t$ for each $t > 0$.*

The concept of quasi-contraction, which is one of the most general contraction criteria, was defined by Ćirić in 1974 as follows.

Definition 2.6. [17] Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping. T is called a quasi-contraction if there exists a $\lambda \in [0, 1)$ such that for all $x, y \in X$ the following inequality holds

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

2.1 Main Results

In this section, we attain some fixed point results related to $(\delta, 1-\delta)$-weak contraction, $(\varphi, 1-\delta)$-weak contraction and Ćirić-type almost contraction in the sense of Berinde defined on G_p-complete G_p-metric space.

First, we provide some definitions which are used to constitute our results.

Definition 2.7. Let (X, G_p) be a G_p-metric space. A mapping $T : X \to X$ is called $(\delta, 1-\delta)$-weak contraction if there exists a $\delta \in (0, 1)$ such that for all $x, y \in X$ the following inequality holds

$$G_p(Tx, Ty, Ty) \leq \delta G_p(x, y, y) + (1-\delta) D_{G_p}(y, Tx). \tag{2.1}$$

Moreover, by G_{p_2} assumption, the $(\delta, 1-\delta)$-weak contraction condition implicitly includes the following dual one

$$G_p(Tx, Ty, Ty) \leq \delta G_p(x, y, y) + (1-\delta) D_{G_p}(x, Ty) \tag{2.2}$$

for all $x, y \in X$. Consequently, to ensure the $(\delta, 1-\delta)$-weak contraction of T, it is necessary to check both (2.1) and (2.2). Thus, by (2.1) and (2.2), $(\delta, 1-\delta)$-weak contraction criteria can be interpreted as the following:

$$G_p(Tx, Ty, Ty) \leq \delta G_p(x, y, y) + (1-\delta)\min\{D_{G_p}(y, Tx), D_{G_p}(x, Ty)\}.$$

Definition 2.8. Let (X, G_p) be a G_p-metric space. A mapping $T : X \to X$ is called $(\varphi, 1-\delta)$-weak contraction if there exist $\delta \in (0, 1)$ and a comparison function φ such that for all $x, y \in X$ the following inequality holds

$$G_p(Tx, Ty, Ty) \leq \varphi(G_p(x, y, y)) + (1-\delta)D_{G_p}(y, Tx). \tag{2.3}$$

Similarly, by (G_{p_2}) assumption, the dual $(\varphi, 1-\delta)$-weak contraction is obtained as the following

$$G_p(Tx, Ty, Ty) \leq \varphi(G_p(x, y, y)) + (1-\delta)D_{G_p}(x, Ty) \tag{2.4}$$

for all $x, y \in X$. Consequently, in order to be regarded as the $(\varphi, 1-\delta)$-weak contraction, a mapping has to satisfy both (2.3) and (2.4). Thus, by integrating (2.3) and (2.4), the $(\varphi, 1-\delta)$-weak contraction condition can be replaced by the following;

$$G_p(Tx, Ty, Ty) \leq \varphi(G_p(x, y, y)) + (1-\delta)\min\{D_{G_p}(y, Tx), D_{G_p}(x, Ty)\}.$$

Theorem 2.6. *Let (X, G_p) be a G_p-complete G_p-metric space and let $T : X \to X$ be a $(\delta, 1-\delta)$-weak contraction mapping. Then T has a unique fixed point in X.*

Proof. $x_0 \in X$ be an arbitrary point. And let for all $n \in \mathbb{N}$ $\{x_n\}$ is defined as $x_n = Tx_{n-1}$. If $x_n = x_{n+1}$ then $x_n = Tx_n$. Thus, the proof is finished. Therefore, let's suppose $x_n \neq x_{n+1}$. Since T is $(\delta, 1-\delta)$-weak contraction, we deduce

$$\begin{aligned} G_p(x_n, x_{n+1}, x_{n+1}) &= G_p(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \delta G_p(x_{n-1}, x_n, x_n) + (1-\delta)D_{G_p}(x_n, Tx_{n-1}) \\ &= \delta G_p(x_{n-1}, x_n, x_n) \end{aligned}$$

where $\delta \in (0, 1)$. Similarly, from (2.1), we obtain

$$G_p(x_{n-1}, x_n, x_n) \leq \delta G_p(x_{n-2}, x_{n-1}, x_{n-1}).$$

By induction, we get

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \delta G_p(x_{n-1}, x_n, x_n) \leq ... \leq \delta^n G_p(x_0, x_1, x_1).$$

Now, let's show $\{x_n\}$ is a G_p-Cauchy sequence. For all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \\ &\quad + G_p(x_{m-1}, x_m, x_m) - [G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\quad + G_p(x_{n+2}, x_{n+2}, x_{n+2}) + \cdots + G_p(x_{m-1}, x_{m-1}, x_{m-1})] \\ &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \\ &\quad + G_p(x_{m-1}, x_m, x_m) \\ &\leq \delta^n G_p(x_0, x_1, x_1) + \delta^{n+1} G_p(x_0, x_1, x_1) + \dots \\ &\quad + \delta^{m-1} G_p(x_0, x_1, x_1) \\ &= \delta^n [1 + \delta + \cdots + \delta^{m-n-1}] G_p(x_0, x_1, x_1) \\ &= \delta^n \frac{1-\delta^{m-n}}{1-\delta} G_p(x_0, x_1, x_1) \\ &\leq \frac{\delta^n}{1-\delta} G_p(x_0, x_1, x_1). \end{aligned}$$

As $n \to \infty$ in the last inequality, we obtain

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0.$$

This shows that $\{x_n\}$ is a G_p-Cauchy sequence in X. Since (X, G_p) is a G_p-complete G_p-metric space, $\{x_n\}$ converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = \lim_{n\to\infty} G_p(x_n, x, x) = G_p(x, x, x) = 0.$$

Therefore, from Lemma 2.4, we have

$$\lim_{n\to\infty} D_{G_p}(x_n, x) = 0.$$

Now, let's show that $G_p(x, Tx, Tx) = 0$. Suppose the contrary. Then $G_p(x, Tx, Tx) > 0$. In this case;

$$\begin{aligned}
G_p(x, Tx, Tx) &\le G_p(x, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, Tx, Tx) - G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\
&\le G_p(x, x_{n+1}, x_{n+1}) + G_p(Tx_n, Tx, Tx) \\
&\le G_p(x, x_{n+1}, x_{n+1}) + \delta G_p(x_n, x, x) + (1-\delta) D_{G_p}(x, Tx_n).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we deduce

$$0 \le G_p(x, Tx, Tx) \le 0$$

which equals to $G_p(x, Tx, Tx) = 0$, that is, $Tx = x$. Hence, T has a fixed point in X. For the uniqueness of the fixed point, suppose y is another fixed point of T but $x \neq y$. Then,

$$\begin{aligned}
G_p(x, y, y) = G_p(Tx, Ty, Ty) &\le \delta G_p(x, y, y) + (1-\delta) D_{G_p}(y, Tx) \\
&\le \delta G_p(x, y, y) + (1-\delta)[G_p(x, y, y) + G_p(y, y, x)] \\
&= G_p(x, y, y) + (1-\delta) G_p(y, y, x) \\
&\le \delta G_p(x, y, y) + (1-\delta) G_p(y, y, x) \\
&\le (\delta + 1 - \delta) \max\{G_p(x, y, y), G_p(y, y, x)\} \\
&= G_p(x, y, y)
\end{aligned}$$

which is a contradiction, so $x = y$ and the uniqueness follows. $\square$

Theorem 2.7. *Let (X, G_p) be a G_p-complete G_p-metric space and $T : X \to X$ be a $(\varphi, 1-\delta)$-weak contraction where φ is a (c)-comparison function, then T has a fixed point in X. Moreover, the fixed point is unique if and only if the (c)-comparison function is given by $\varphi(t) = \delta t$, where $\delta \in (0, 1)$.*

Proof. $x_0 \in X$ be an arbitrary point. And let for all $n \in \mathbb{N}$, $\{x_n\}$ is defined as $x_n = Tx_{n-1}$. If $x_n = x_{n+1}$ then $x_n = Tx_n$. So, the proof is finished. Therefore, let's suppose $x_n \neq x_{n+1}$. Since T is a $(\varphi, 1-\delta)$-weak contraction, we have the following

$$\begin{aligned}
G_p(x_n, x_{n+1}, x_{n+1}) &= G_p(Tx_{n-1}, Tx_n, Tx_n) \\
&\le \varphi(G_p(x_{n-1}, x_n, x_n)) + (1-\delta) D_{G_p}(x_n, Tx_{n-1}) \\
&= \varphi(G_p(x_{n-1}, x_n, x_n)).
\end{aligned}$$

Similarly, from (2.3), we deduce

$$G_p(x_{n-1}, x_n, x_n) \le \varphi(G_p(x_{n-2}, x_{n-1}, x_{n-1})).$$

By induction, we obtain

$$G_p(x_n, x_{n+1}, x_{n+1}) \le \varphi(G_p(x_{n-1}, x_n, x_n)) \le \cdots \le \varphi^n(G_p(x_0, x_1, x_1)).$$

Now, let's show $\{x_n\}$ is a G_p-Cauchy sequence. For all $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \\ &\quad + G_p(x_{m-1}, x_m, x_m) - [G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\quad + G_p(x_{n+2}, x_{n+2}, x_{n+2}) + \dots + G_p(x_{m-1}, x_{m-1}, x_{m-1})] \\ &= \sum_{k=n}^{m-1} G_p(x_k, x_{k+1}, x_{k+1}) - \sum_{k=n}^{m-2} G_p(x_{k+1}, x_{k+1}, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} G_p(x_k, x_{k+1}, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \varphi^k(G_p(x_0, x_1, x_1)) \\ &\leq \sum_{k=n}^{\infty} \varphi^k(G_p(x_0, x_1, x_1)). \end{aligned}$$

Since φ is a (c)-comparison function, $\sum_{k=0}^{\infty} \varphi^k(G_p(x_0, x_1, x_1))$ is convergent and as $n \to \infty$ we obtain $\varphi^k(G_p(x_0, x_1, x_1)) \to 0$. Therefore, $\lim_{n\to\infty} G_p(x_n, x_m, x_m) = 0$ which implies $\{x_n\}$ sequence is a G_p-Cauchy sequence in X. Since (X, G_p) is a G_p-complete G_p-metric space, the sequence $\{x_n\}$ converges to a point $x \in X$ such as

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = \lim_{n\to\infty} G_p(x_n, x, x) = G_p(x, x, x) = 0. \tag{2.5}$$

So, from Lemma 2.4,

$$\lim_{n\to\infty} D_{G_p}(x_n, x) = 0.$$

Now, claim that $G_p(x, Tx, Tx) = 0$. Suppose the contrary, that is, $G_p(x, Tx, Tx) > 0$. In this case, from (2.5), there exists an $n_0 \in \mathbb{N}$ such that

$$G_p(x_n, x, x) < \frac{G_p(x, Tx, Tx)}{2}.$$

Then, by using Lemma 2.5, we obtain

$$\begin{aligned} G_p(x, Tx, Tx) &\leq G_p(x, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, Tx, Tx) - G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\leq G_p(x, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, Tx, Tx) \\ &= G_p(x, x_{n+1}, x_{n+1}) + G_p(Tx_n, Tx, Tx) \\ &\leq G_p(x, x_{n+1}, x_{n+1}) + \varphi(G_p(x_n, x, x)) + (1-\delta)D_{G_p}(x, Tx_n) \\ &\leq G_p(x, x_{n+1}, x_{n+1}) + \varphi\left(\frac{G_p(x, Tx, Tx)}{2}\right) + (1-\delta)D_{G_p}(x, x_{n+1}) \\ &< G_p(x, x_{n+1}, x_{n+1}) + \frac{G_p(x, Tx, Tx)}{2} + (1-\delta)D_{G_p}(x, x_{n+1}). \end{aligned}$$

If we take the limit of last inequality as $n \to \infty$, we deduce

$$G_p(x, Tx, Tx) < \frac{G_p(x, Tx, Tx)}{2}$$

which is a contradiction. Therefore, as our assumption being false, we obtain $Tx = x$. Hence, T has a fixed point in X. For the uniqueness of the fixed point, suppose y is another fixed point of

T. If $G_p(x,y,y)=0$, then $x=y$ is clear. So we assume, $G_p(x,y,y)>0$. Now observe we have the following

$$\begin{aligned}
G_p(x,y,y)=G_p(Tx,Ty,Ty) &\leq \varphi(G_p(x,y,y))+(1-\delta)D_{G_p}(y,Tx)\\
&\leq \delta G_p(x,y,y)+(1-\delta)[G_p(x,y,y)+G_p(y,y,x)]\\
&= G_p(x,y,y)+(1-\delta)G_p(y,y,x)\\
&\leq \delta G_p(x,y,y)+(1-\delta)G_p(y,y,x)\\
&\leq (\delta+1-\delta)\max\{G_p(x,y,y),G_p(y,y,x)\}\\
&= G_p(x,y,y)
\end{aligned}$$

which is a contradiction. Thus, $x=y$ and the uniqueness follows. □

Theorem 2.8. *Let (X,G_p) be a G_p-complete G_p-metric space and $T:X\to X$ be a Ćirić $(\alpha,1-\alpha)$-weak contraction, that is, there exists $\alpha\in\left[0,\frac{1}{2}\right)$ such that for all $x,y\in X$ the following holds*

$$\begin{aligned}
G_p(Tx,Ty,Ty) \leq\ & \alpha M(x,y,y)+(1-\alpha)\min\{D_{G_p}(x,Tx),D_{G_p}(y,Ty),D_{G_p}(x,Ty),\\
& D_{G_p}(y,Tx)\}
\end{aligned}\tag{2.6}$$

where

$$M(x,y,y) = \max\{G_p(x,y,y),G_p(x,Tx,Tx),G_p(y,Ty,Ty),G_p(x,Ty,Ty),G_p(y,Tx,Tx)\}.$$

Then, T has a unique fixed point in X.

Proof. $x_0\in X$ be an arbitrary point. And let for all $n\in\mathbb{N}$, $\{x_n\}$ is defined as $x_n=Tx_{n-1}$. If $x_n=x_{n+1}$, then $x_n=Tx_n$. So, the proof is completed. Therefore, let's suppose $x_n\neq x_{n+1}$. From (2.6), we obtain

$$\begin{aligned}
G_p(x_n,x_{n+1},x_{n+1}) &= G_p(Tx_{n-1},Tx_n,Tx_n)\\
&\leq \alpha M(x_{n-1},x_n,x_n)+(1-\alpha)\min\{D_{G_p}(x_{n-1},Tx_{n-1}),\\
&\quad D_{G_p}(x_n,Tx_n),D_{G_p}(x_{n-1},Tx_n),D_{G_p}(x_n,Tx_{n-1})\}\\
&= \alpha M(x_{n-1},x_n,x_n)+(1-\alpha)\min\{D_{G_p}(x_{n-1},x_n),\\
&\quad D_{G_p}(x_n,x_{n+1}),D_{G_p}(x_{n-1},x_{n+1}),D_{G_p}(x_n,x_n)\}\\
&= \alpha M(x_{n-1},x_n,x_n),
\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}
M(x_{n-1},x_n,x_n) &= \max\{G_p(x_{n-1},x_n,x_n),G_p(x_{n-1},Tx_{n-1},Tx_{n-1}),\\
&\quad G_p(x_n,Tx_n,Tx_n),G_p(x_{n-1},Tx_n,Tx_n),\\
&\quad G_p(x_n,Tx_{n-1},Tx_{n-1})\}\\
&= \max\{G_p(x_{n-1},x_n,x_n),G_p(x_n,x_{n+1},x_{n+1}),\\
&\quad G_p(x_{n-1},x_{n+1},x_{n+1})\}.
\end{aligned}$$

If $M(x_{n-1},x_n,x_n)=G_p(x_n,x_{n+1},x_{n+1})$ then from (2.7), we deduce

$$G_p(x_n,x_{n+1},x_{n+1})\leq\alpha G_p(x_n,x_{n+1},x_{n+1})<G_p(x_n,x_{n+1},x_{n+1})$$

which is a contradiction.

If $M(x_{n-1},x_n,x_n)=G_p(x_{n-1},x_n,x_n)$, then we get

$$G_p(x_n,x_{n+1},x_{n+1})\leq\alpha G_p(x_{n-1},x_n,x_n).\tag{2.8}$$

And lastly, if $M(x_{n-1}, x_n, x_n) = G_p(x_{n-1}, x_{n+1}, x_{n+1})$, then we obtain

$$\begin{aligned} G_p(x_n, x_{n+1}, x_{n+1}) &\leq \alpha G_p(x_{n-1}, x_{n+1}, x_{n+1}) \\ &\leq \alpha[G_p(x_{n-1}, x_n, x_n) + G_p(x_n, x_{n+1}, x_{n+1}) \\ &\quad - G_p(x_n, x_n, x_n)] \\ &\leq \alpha[G_p(x_{n-1}, x_n, x_n) + G_p(x_n, x_{n+1}, x_{n+1})] \end{aligned}$$

and consequently, we infer

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \frac{\alpha}{1-\alpha} G_p(x_{n-1}, x_n, x_n). \tag{2.9}$$

Considering $h = \frac{\alpha}{1-\alpha}$, we conclude that $h \in [0, 1)$ and so, we obtain

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq hG_p(x_{n-1}, x_n, x_n). \tag{2.10}$$

Therefore, by taking a constant r as $r \in \{\alpha, h\}$ and using (2.8) and (2.10), we deduce the following inequality

$$G_p(x_n, x_{x+1}, x_{n+1}) \leq rG_p(x_{n-1}, x_n, x_n).$$

By induction, we acquire

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq r^n G_p(x_0, x_1, x_1).$$

Now, let's show $\{x_n\}$ is a G_p-Cauchy sequence. For all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \\ &\quad + G_p(x_{m-1}, x_m, x_m) \\ &\leq r^n G_p(x_0, x_1, x_1) + r^{n+1} G_p(x_0, x_1, x_1) + \dots \\ &\quad + r^{m-1} G_p(x_0, x_1, x_1) \\ &= r^n [1 + r + \dots + r^{m-n-1}] G_p(x_0, x_1, x_1) \\ &\leq r^n \frac{1 - r^{m-n}}{1-r} G_p(x_0, x_1, x_1) \\ &\leq \frac{r^n}{1-r} G_p(x_0, x_1, x_1). \end{aligned}$$

Taking the limit in the last inequality as $n \to \infty$, we get

$$\lim_{n,m \to \infty} G_p(x_n, x_m, x_m) = 0.$$

Thus, $\{x_n\}$ is a G_p-Cauchy sequence in X. Since (X, G_p) is a G_p-complete G_p-metric space, $\{x_n\}$ converges to a point $x \in X$ such that

$$\lim_{n,m \to \infty} G_p(x_n, x_m, x_m) = \lim_{n \to \infty} G_p(x_n, x, x) = G_p(x, x, x) = 0. \tag{2.11}$$

Moreover, from Lemma 2.4, we have

$$\lim_{n \to \infty} D_{G_p}(x_n, x) = 0.$$

Now let's show $x = Tx$. Suppose the contrary, that is, $x \neq Tx$. Then, from (2.11), we deduce

$$\begin{aligned} G_p(x, Tx, Tx) &\leq G_p(x, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, Tx, Tx) - \\ &\quad G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\leq G_p(x, x_{n+1}, x_{n+1}) + G_p(Tx_n, Tx, Tx) \\ &\leq G_p(x, x_{n+1}, x_{n+1}) + \alpha M(x_n, x, x) \\ &\quad + (1-\alpha) \min\{D_{G_p}(x_n, Tx_n), D_{G_p}(x, Tx), D_{G_p}(x_n, Tx), D_{G_p}(x, Tx_n)\} \\ &\leq G_p(x, x_{n+1}, x_{n+1}) + \alpha M(x_n, x, x) + (1-\alpha) \min\{D_{G_p}(x_n, x_{n+1}), \\ &\quad D_{G_p}(x, Tx), D_{G_p}(x_n, Tx), D_{G_p}(x, x_{n+1})\} \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} M(x_n, x, x) &= \max\{G_p(x_n, x, x), G_p(x_n, Tx_n, Tx_n), \\ &\quad G_p(x, Tx, Tx), G_p(x_n, Tx, Tx), G_p(x, Tx_n, Tx_n)\} \\ &= \max\{G_p(x_n, x, x), G_p(x_n, x_{n+1}, x_{n+1}), G_p(x, Tx, Tx), \\ &\quad G_p(x_n, Tx, Tx), G_p(x, x_{n+1}, x_{n+1}\}. \end{aligned} \tag{2.13}$$

Letting $n \to \infty$ in (2.13), we deduce

$$\lim_{n\to\infty} M(x_n, x, x) = G_p(x, Tx, Tx).$$

Consequently, by taking the limit of (2.12) as $n \to \infty$, we acquire

$$G_p(x, Tx, Tx) \leq \alpha G_p(x, Tx, Tx)$$

which is a contradiction. Thus, x is a fixed point of T in X. For the uniqueness of the fixed point, suppose y is another fixed point of T but $x \neq y$. Then,

where

$$\begin{aligned} M(x, y, y) &= \max\{G_p(x, y, y), G_p(x, Tx, Tx), G_p(y, Ty, Ty), G_p(x, Ty, Ty), \\ &\quad G_p(y, Tx, Tx)\} \\ &= G_p(x, y, y). \end{aligned}$$

As a result,

$$G_p(x, y, y) \leq \alpha G_p(x, y, y) < G_p(x, y, y)$$

□

which is impossible. So, $x = y$ and the uniqueness follows.

Theorem 2.9. *Let (X, G_p) be a G_p-complete G_p-metric space and let $T, S : X \to X$ be two $(\delta, 1-\delta)$-weak contractions, that is, there exists a $\delta \in (0, 1)$ such that for all $x, y \in X$ the following holds*

$$G_p(Tx, Sy, Sy) \leq \delta G_p(x, y, y) + (1-\delta)\min\{D_{G_p}(y, Tx), D_{G_p}(x, Sy)\}. \tag{2.14}$$

Then T and S have a common fixed point in X.

Proof. $x_0 \in X$ be an arbitrary point and choose $x_1 = Tx_0$ and $x_2 = Sx_1$. Continuing this process, we construct a sequence $\{x_n\}$ in X such that for each $n \geq 0$,

$$x_{2n+1} = Tx_{2n} \quad \textit{and} \quad x_{2n+2} = Sx_{2n+1}.$$

Suppose $G_p(x_n, x_{n+1}, x_{n+1}) = 0$ for some $n \in \mathbb{N}$. Without loss of generality, we assume $n = 2k$ for some $k \in \mathbb{N}$. Thus, $G_p(x_{2k}, x_{2k+1}, x_{2k+1}) = 0$ implies $x_{2k} = x_{2k+1}$. Now, claim that $G_p(x_{2k+1}, x_{2k+2}, x_{2k+2}) > 0$. Hence, by using (2.14), we obtain

$$\begin{aligned} G_p(x_{2k+1}, x_{2k+2}, x_{2k+2}) &= G_p(Tx_{2k}, Sx_{2k+1}, Sx_{2k+1}) \\ &\leq \delta G_p(x_{2k}, x_{2k+1}, x_{2k+1}) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2k+1}, Tx_{2k}), D_{G_p}(x_{2k}, Sx_{2k+1})\} \\ &= \delta G_p(x_{2k}, x_{2k+1}, x_{2k+1}) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2k+1}, x_{2k+1}), D_{G_p}(x_{2k}, x_{2k+2})\} \\ &= 0. \end{aligned}$$

Therefore, we conclude that $x_{2k+1} = x_{2k+2}$. Hence, we have $x_{2k} = Tx_{2k} = Sx_{2k}$ which implies that x_{2k} is a common fixed point of T and S. As a result, we can also suppose that the successive terms of $\{x_n\}$ are different. Thus, we get $G_p(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. From (2.14), we have

$$\begin{aligned} G_p(x_{2t}, x_{2t+1}, x_{2t+1}) &= G_p(x_{2t+1}, x_{2t}, x_{2t}) = G_p(Tx_{2t}, Sx_{2t-1}, Sx_{2t-1}) \\ &\leq \delta G_p(x_{2t}, x_{2t-1}, x_{2t-1}) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2t-1}, Tx_{2t}), D_{G_p}(x_{2t}, Sx_{2t-1})\} \\ &= \delta G_p(x_{2t}, x_{2t-1}, x_{2t-1}) \\ &= \delta G_p(x_{2t-1}, x_{2t}, x_{2t}). \end{aligned} \tag{2.15}$$

If n is odd, then $n = 2t+1$ for some $t \in \mathbb{N}$ and so we deduce

$$\begin{aligned} G_p(x_{2t+1}, x_{2t+2}, x_{2t+2}) &= G_p(Tx_{2t}, Sx_{2t+1}, Sx_{2t+1}) \\ &\leq \delta G_p(x_{2t}, x_{2t+1}, x_{2t+1}) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2t+1}, Tx_{2t}), D_{G_p}(x_{2t}, Sx_{2t+1})\} \\ &= \delta G_p(x_{2t}, x_{2t+1}, x_{2t+1}). \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we have

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \delta G_p(x_{n-1}, x_n, x_n).$$

By induction,

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \delta^n G_p(x_0, x_1, x_1).$$

Now let's show that $\{x_n\}$ is a G_p-Cauchy sequence. For all $m, n \in \mathbb{N}$ with $m > n$, we acquire

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \\ &\quad +G_p(x_{m-1}, x_m, x_m) \\ &\leq \delta^n G_p(x_0, x_1, x_1) + \delta^{n+1} G_p(x_0, x_1, x_1) + \dots \\ &\quad +\delta^{m-1} G_p(x_0, x_1, x_1) \\ &= \delta^n [1 + \delta + \dots + \delta^{m-n-1}] G_p(x_0, x_1, x_1) \\ &= \delta^n \frac{1-\delta^{m-n}}{1-\delta} G_p(x_0, x_1, x_1) \\ &\leq \frac{\delta^n}{1-\delta} G_p(x_0, x_1, x_1). \end{aligned}$$

By taking the limit of last inequality as $n \to \infty$, we deduce

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0$$

which implies $\{x_n\}$ is a G_p-Cauchy sequence in X. Since (X, G_p) is a G_p-complete G_p-metric space, $\{x_n\}$ converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = \lim_{n\to\infty} G_p(x_n, x, x) = G_p(x, x, x) = 0.$$

Furthermore, from Lemma 2.4, we have

$$\lim_{n\to\infty} D_{G_p}(x_n, x) = 0.$$

Now, let's show that $G_p(Tx,x,x)=0$. Suppose the contrary, that is, $G_p(Tx,x,x)>0$. Then, from (2.14) and triangle inequality, we have

$$\begin{aligned}
G_p(Tx,x,x) &\leq G_p(Tx,x_{2n+2},x_{2n+2})+G_p(x_{2n+2},x,x)\\
&\quad -G_p(x_{2n+2},x_{2n+2},x_{2n+2})\\
&\leq G_p(Tx,x_{2n+2},x_{2n+2})+G_p(x_{2n+2},x,x)\\
&= G_p(Tx,Sx_{2n+1},Sx_{2n+1})+G_p(x_{2n+2},x,x)\\
&\leq \delta G_p(x,x_{2n+1},x_{2n+1})\\
&\quad +(1-\delta)\min\{D_{G_p}(x_{2n+1},Tx),D_{G_p}(x,Sx_{2n+1})\}+G_p(x_{2n+2},x,x).
\end{aligned}$$

As $n\to\infty$ in the last inequality, we obtain $G_p(Tx,x,x)=0$ indicating $Tx=x$. Now, let's show $Sx=x$. Suppose $Sx\neq x$. Then, by (2.14),

$$\begin{aligned}
G_p(x,Sx,Sx) &= G_p(Tx,Sx,Sx)\\
&\leq \delta G_p(x,x,x)+(1-\delta)\min\{D_{G_p}(x,Sx),D_{G_p}(x,Tx)\}\\
&= 0.
\end{aligned}$$

Taking the limit in the last inequality as $n\to\infty$, we derive $G_p(x,Sx,Sx)=0$ which denotes $Sx=x$. Therefore, $Tx=x=Sx$ and so x is a common fixed point of T and S in X. □

Corollary 2.10. *Let (X,G_p) be a G_p-complete G_p-metric space and let $T,S:X\to X$ be two $(\delta,1-\delta)$-weak contractions, that is, there exists a $\delta\in(0,1)$ such that for all $x,y\in X$ the following holds*

$$G_p(Tx,Sy,Sy)\leq \delta G_p(x,y,y)+(1-\delta)\min\{D_{G_p}(y,Tx),D_{G_p}(x,Sy),D_{G_p}(x,Tx)\}.$$

Then T and S have a unique common fixed point in X.

Theorem 2.11. *Let (X,G_p) be a G_p-complete G_p-metric space and let $T,S:X\to X$ be two $(\varphi,1-\delta)$-weak contractions, that is, there exist $\delta\in(0,1)$ and a (c)-comparison function such that for all $x,y\in X$ the following holds*

$$G_p(Tx,Sy,Sy)\leq \varphi(G_p(x,y,y))+(1-\delta)\min\{D_{G_p}(y,Tx),D_{G_p}(x,Sy)\}. \tag{2.17}$$

Then T and S have a common fixed point in X.

Proof. $x_0\in X$ be an arbitrary point and choose $x_1=Tx_0$ and $x_2=Sx_1$. Continuing this process we construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1}=Tx_{2n}\quad \textit{and}\quad x_{2n+2}=Sx_{2n+1} \tag{2.18}$$

for each $n\geq 0$. Suppose $G_p(x_n,x_{n+1},x_{n+1})=0$ for some $n\in\mathbb{N}$. Without loss of generality, we assume $n=2k$ for some $k\in\mathbb{N}$. Thus, $G_p(x_{2k},x_{2k+1},x_{2k+1})=0$ which implies $x_{2k}=x_{2k+1}$. Now assume $G_p(x_{2k+1},x_{2k+2},x_{2k+2})>0$. Hence, using (2.17), we obtain

$$\begin{aligned}
G_p(x_{2k+1},x_{2k+2},x_{2k+2}) &= G_p(Tx_{2k},Sx_{2k+1},Sx_{2k+1})\\
&\leq \varphi(G_p(x_{2k},x_{2k+1},x_{2k+1}))\\
&\quad +(1-\delta)\min\{D_{G_p}(x_{2k+1},Tx_{2k}),D_{G_p}(x_{2k},Sx_{2k+1})\}\\
&= \varphi(0)\\
&\leq \varphi(G_p(x_{2k+1},x_{2k+2},x_{2k+2}))\\
&< G_p(x_{2k+1},x_{2k+2},x_{2k+2})
\end{aligned}$$

which is a contradiction. Then, we conclude that $x_{2k} = x_{2k+1} = x_{2k+2}$. So, we deduce $x_{2k} = Tx_{2k} = Sx_{2k}$ proving x_{2k} is a common fixed point of T and S. Therefore, we can claim that the successive terms of $\{x_n\}$ are different. Then $G_p(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. Then, by using (2.17), we obtain

$$\begin{aligned} G_p(x_{2t}, x_{2t+1}, x_{2t+1}) &= G_p(x_{2t+1}, x_{2t}, x_{2t}) = G_p(Tx_{2t}, Sx_{2t-1}, Sx_{2t-1}) \\ &\leq \varphi(G_p(x_{2t}, x_{2t-1}, x_{2t-1})) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2t-1}, Tx_{2t}), D_{G_p}(x_{2t}, Sx_{2t-1})\} \\ &= \varphi(G_p(x_{2t}, x_{2t-1}, x_{2t-1})) \\ &= \varphi(G_p(x_{2t-1}, x_{2t}, x_{2t})). \end{aligned} \tag{2.19}$$

If n is odd, then $n = 2t+1$ for some $t \in \mathbb{N}$, so we have

$$\begin{aligned} G_p(x_{2t+1}, x_{2t+2}, x_{2t+2}) &= G_p(Tx_{2t}, Sx_{2t+1}, Sx_{2t+1}) \\ &\leq \varphi(G_p(x_{2t}, x_{2t+1}, x_{2t+1})) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2t+1}, Tx_{2t}), D_{G_p}(x_{2t}, Sx_{2t+1})\} \\ &= \varphi(G_p(x_{2t}, x_{2t+1}, x_{2t+1})). \end{aligned} \tag{2.20}$$

From (2.19) and (2.20), we get

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \varphi(G_p(x_{n-1}, x_n, x_n)).$$

By induction, we obtain

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \varphi^n(G_p(x_0, x_1, x_1)).$$

Now let's show that $\{x_n\}$ is a G_p-Cauchy sequence. For all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \\ &\quad +G_p(x_{m-1}, x_m, x_m) \\ &\leq \varphi^n(G_p(x_0, x_1, x_1)) + \varphi^{n+1}(G_p(x_0, x_1, x_1)) + \dots \\ &\quad +\varphi^{m-1}(G_p(x_0, x_1, x_1)) \\ &= \sum_{i=n}^{m-1} \varphi^i(G_p(x_0, x_1, x_1)) \\ &\leq \sum_{i=n}^{\infty} \varphi^i(G_p(x_0, x_1, x_1)). \end{aligned}$$

In the last inequality as $n \to \infty$, we get

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0.$$

Thus, $\{x_n\}$ is a G_p-Cauchy sequence in X. Since (X, G_p) is a G_p-complete G_p-metric space, $\{x_n\}$ converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = \lim_{n\to\infty} G_p(x_n, x, x) = G_p(x, x, x) = 0$$

Furthermore, from Lemma 2.4,

$$\lim_{n\to\infty} D_{G_p}(x_n, x) = 0.$$

Now let's show $G_p(Tx, x, x) = 0$. Suppose the contrary, that is, $G_p(Tx, x, x) > 0$. Then, from (2.17) for all $n \geq n_0$, there exists an $n_0 \in \mathbb{N}$ such as

$$G_p(x, x_{2n+1}, x_{2n+1}) < \frac{G_p(Tx, x, x)}{2}.$$

Consequently, we obtain

$$\begin{aligned} G_p(Tx,x,x) &\leq G_p(Tx,x_{2n+2},x_{2n+2})+G_p(x_{2n+2},x,x) \\ &\quad -G_p(x_{2n+2},x_{2n+2},x_{2n+2}) \\ &\leq G_p(Tx,x_{2n+2},x_{2n+2})+G_p(x_{2n+2},x,x) \\ &= G_p(Tx,Sx_{2n+1},Sx_{2n+1})+G_p(x_{2n+2},x,x) \\ &\leq \varphi(G_p(x,x_{2n+1},x_{2n+1})) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2n+1},Tx),D_{G_p}(x,Sx_{2n+1})\}+G_p(x_{2n+2},x,x) \\ &\leq \varphi\left(\frac{G_p(Tx,x,x)}{2}\right) \\ &\quad +(1-\delta)\min\{D_{G_p}(x_{2n+1},Tx),D_{G_p}(x,x_{2n+2})\}+G_p(x_{2n+2},x,x). \end{aligned}$$

Letting $n\to\infty$ in the last inequality and using Lemma 2.5, we get

$$G_p(Tx,x,x)\leq\varphi\left(\frac{G_p(Tx,x,x)}{2}\right)<\frac{G_p(Tx,x,x)}{2}$$

which is a contradiction. Therefore, $G_p(Tx,x,x)=0$ indicating $Tx=x$. Now, let's show $Sx=x$. Assume that $Sx\neq x$. Then by (2.17), we deduce

$$\begin{aligned} G_p(Sx,x,x) &= G_p(Tx,Sx,Sx) \\ &\leq \varphi(G_p(x,x,x))+(1-\delta)\min\{D_{G_p}(x,Tx),D_{G_p}(x,Sx)\} \\ &= \varphi(0) \\ &\leq \varphi(G_p(x,Sx,Sx)) \\ &< G_p(x,Sx,Sx) \end{aligned}$$

which is a contradiction. Hence, $Sx=x$. As a result $Tx=x=Sx$ and so x is a common fixed point of T and S in X. □

Corollary 2.12. *Let (X,G_p) be a G_p-complete G_p-metric space and let $T,S:X\to X$ be two $(\varphi,1-\delta)$-weak contractions, that is, there exist $\delta\in(0,1)$ and a (c)-comparison function such that for all $x,y\in X$ the following holds*

$$G_p(Tx,Sy,Sy)\leq\varphi(G_p(x,y,y))+(1-\delta)\min\{D_{G_p}(y,Tx),D_{G_p}(x,Sy),D_{G_p}(x,Tx)\}.$$

Then T and S have a unique common fixed point in X.

Theorem 2.13. *Let (X,G_p) be a G_p-complete G_p-metric space and let $T,S:X\to X$ be two strong Ćirić $(\alpha,1-\alpha)$-weak contractions, that is, there exists a constant $\alpha\in[0,\frac{1}{2})$ such that for all $x,y\in X$ the following holds*

$$G_p(Tx,Sy,Sy)\leq\alpha M(x,y,y)+(1-\alpha)\min\{D_{G_p}(y,Tx),D_{G_p}(x,Sy)\} \tag{2.21}$$

where

$$M(x,y,y)=\max\{G_p(x,y,y),G_p(x,Tx,Tx),G_p(y,Sy,Sy),G_p(x,Sy,Sy),G_p(y,Tx,Tx)\}.$$

Then T and S have a common fixed point in X.

Corollary 2.14. *Let (X,G_p) be a G_p-complete G_p-metric space and let $T,S:X\to X$ be two Ćirić $(\alpha,1-\alpha)$-weak contractions, that is, there exists a constant $\alpha\in[0,\frac{1}{2})$ such that for all $x,y\in X$ the following holds*

$$G_p(Tx,Sy,Sy)\leq\alpha M(x,y,y)+(1-\alpha)\min\{D_{G_p}(y,Tx),D_{G_p}(x,Sy),D_{G_p}(x,Tx)\}$$

where

$$M(x,y,y)=\max\{G_p(x,y,y),G_p(x,Tx,Tx),G_p(y,Sy,Sy),G_p(x,Sy,Sy),G_p(y,Tx,Tx)\}.$$

Then T and S have a unique common fixed point in X.

3 Examples

In this section, we present some examples to illustrate the usability of the previously obtained results.

Example 3.1. *Let $X = [0,1]$, $G_p : X \times X \times X \to [0,\infty)$ be defined by $G_p(x,y,z) = \max\{x,y,z\}$. Then (X, G_p) is a G_p-complete G_p-metric space and for all $x, y \in X$*

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y) = |x-y|.$$

Define $T : X \to X$ as $Tx = \dfrac{x}{5}$ for all $x, y \in X$. Without loss of generality suppose $x \leq y$ for all $x, y \in X$ and $\delta = \dfrac{2}{5}$. Then, we get

$$\begin{aligned} G_p(Tx,Ty,Ty) = \max\left\{\frac{x}{5}, \frac{y}{5}\right\} &= \frac{y}{5} \\ &\leq \frac{2y}{5} = \delta \max\{x,y\} \\ &\leq \delta \max\{x,y\} + (1-\delta)\min\left\{\left|x - \frac{y}{5}\right|, \left|y - \frac{x}{5}\right|\right\} \\ &= \delta G_p(x,y,y) + (1-\delta)\min\{D_{G_p}(x,Ty), \\ &\quad D_{G_p}(y,Tx)\}. \end{aligned}$$

Therefore all the conditions of (2.1) and (2.2) hold and T has a fixed point in X.

Example 3.2. *Let $X = [0,1]$, $G_p : X \times X \times X \to [0,\infty)$ be defined by $G_p(x,y,z) = \max\{x,y,z\}$. Then (X, G_p) is a G_p-complete G_p-metric space and for all $x, y \in X$*

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y) = |x-y|.$$

Define $T : X \to X$ and $\varphi : [0,\infty] \to [0,\infty]$ respectively as $Tx = \dfrac{x}{3}$ and $\varphi = \dfrac{2t}{3}$ for all $x, y \in X$. Without loss of generality suppose $x \leq y$, $\delta \in (0,1)$ and φ is a comparison function. Then, we get

$$\begin{aligned} G_p(Tx,Ty,Ty) = \max\left\{\frac{x}{3}, \frac{y}{3}\right\} &= \frac{y}{3} \\ &\leq \frac{2y}{3} = \varphi(y) = \varphi(\max\{x,y\}) \\ &\leq \varphi(\max\{x,y\}) + (1-\delta)\min\left\{\left|x - \frac{y}{3}\right|, \left|y - \frac{x}{3}\right|\right\} \\ &= \varphi(G_p(x,y,y)) + (1-\delta)\min\{D_{G_p}(x,Ty), \\ &\quad D_{G_p}(y,Tx)\}. \end{aligned}$$

Therefore, all the conditions of (2.3) and (2.4) hold and T has a fixed point in X.

Example 3.3. *Let $X = [0,1]$, $G_p : X \times X \times X \to [0,\infty)$ be defined by $G_p(x,y,z) = \max\{x,y,z\}$. Then (X, G_p) is a G_p-complete G_p-metric space and for all $x, y \in X$*

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y) = |x-y|.$$

Define $T, S : X \to X$ as $Tx = \dfrac{x}{6}$ and $Sx = \dfrac{x}{2}$. Also $\varphi : [0,\infty] \to [0,\infty]$ is defined as $\varphi = \dfrac{3t}{4}$. Without loss of generality suppose $x \leq y$, $\delta \in (0,1)$ and φ is a comparison function. Then, we *obtain*

$$\begin{aligned} G_p(Tx,Sy,Sy) = \max\left\{\frac{x}{6},\frac{y}{2}\right\} &= \frac{y}{2} \\ &\leq \frac{3y}{4} = \varphi(y) = \varphi(\max\{x,y\}) \\ &\leq \varphi(\max\{x,y\}) + (1-\delta)\min\left\{\left|x-\frac{y}{2}\right|,\left|y-\frac{x}{6}\right|\right\} \\ &= \varphi(G_p(x,y,y)) + (1-\delta)\min\{D_{G_p}(x,Sy), \\ &\quad D_{G_p}(y,Tx)\}. \end{aligned}$$

Then, the conditions of (2.17) hold and T and S have a common fixed point in X.

4 Conclusions

In the present paper, we define the concepts of $(\delta, 1-\delta)$-weak contraction, $(\varphi, 1-\delta)$-weak contraction and Ćirić-type almost contraction in the sense of Berinde in G_p-complete G_p-metric space and establish some fixed point theorems in G_p-metric space which demonstrate the existence of fixed points and common fixed points of mappings satisfying Berinde-type contractions. We note that the results of this paper generalize several results in the literature.

Competing Interests

Authors have declared that no competing interests exist.

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