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Second derivative Hybrid Block Backward Differentiation Formulae for Numerical Solution of Stiff Systems

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Authors' contributions

This work was carried out in collaboration between all authors. This work was carried out in collaboration between all authors. Author GMK derived computational method. Author YS analyzed the basic properties of the computational method while author IAB implemented the method with aid of MAPLE SOFTWARE. All authors read and approved the final manuscript.

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Abstract

In this research work, two and three step second derivative hybrid block backward differentiation formulae (SDHBBDF) method of order 6 and 7 are presented for the numerical solution of stiff initial value problems. The block scheme was obtained through increasing the number k in the multi-step collocation (MC) with the aid of maple software. The stability analysis of the method have shown that the schemes are A-stable and consistent. We compared SDHBBDF methods with exact solutions and have shown that the results obtained using proposed new block methods are excellent for the solution of stiff problems.

Keywords: Hybrid; block method; backward differentiation formula; second derivative; stiff systems.

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1 Introduction

We consider an approximation of the solution of first order differential equations of the form

$$y'(x) = f(x, y), (a \leq x \leq b), y(x_0) = y_0 \tag{1.1}$$

It also known that (1.1) is better handled by schemes with larger stability intervals. In particular, A- stable methods are of great importance. However, for every large systems arising from the semi discretization of parabolic PDEs, A- stable methods converges very slowly to the exact solution. The development of continuous method has been the subject of growing interest due to the fact that continuous methods enjoy certain advantages, such as the potential for them to provide defect control by Enright [1]. The block method produced numerical solutions with less than computational effort as to compare to non-block method by Majid [2]. Block methods have been considered by various authors among who are A-stable implicit one block methods with higher orders by Shampine and Watts [3]. Voss and Abbas [4] proposed one-step fourth-order block method and it was shown that the method can be paralleled as further research to enhance the efficiency. In this research, the block method (SDHBBDF) are constructed using multi- step collocation approach proposed by Lie and Norsett [5] and referred to as the block methods by Onumanyi et al. [6]. The block methods are seen as a set of linear multi-step methods which are applied simultaneously yielding more approximation.

2 Derivation of Methods

In this section, the derivation techniques of three step block methods for solution of stiff systems of initial value problems. In this regard, the derivation will carry out through the interpolation and collocation on the equi-distance step points.

Let the approximate solution be given as power series of a single variable x in the form

$$y(x) = \sum_{i=0}^{p-1} \alpha_i x^i \tag{1.2}$$

Which is twice continuously differentiable function of y(x).We set the sum $r + S + t$ to be equal to p so as to be able to determine $\{\alpha_i\}$ uniquely. Our aim is to utilize not only the interpolation point $\{x_i\}$ but also several collocation point on the interpolation interval and to fit $y(x)$ for $y'(x)$ and $y''(x)$ Yakubu et al. [7]. We show the following conditions,

$$y(x_{n+j}) = y_{n+j}, \ (j = 0.1 \dots\dots r - 1) \tag{1.3}$$

$$y'(c_{n+j}) = \bar{y}'_{n+j}, \ (j = 0.1 \dots\dots s - 1) \tag{1.4}$$

$$y''(c_{n+j}) = \bar{y}''_{n+j}, \ (j = 0.1 \dots\dots t - 1) \tag{1.5}$$

Where $\{c_{n+j}\}$ are collocation points distributed on the step-points array. y_{n+j} is the interpolation data of $y(x)$ on x_{n+j} and $\bar{y}'_{n+j}$, $\bar{y}''_{n+j}$ are the collocation data of $y'(x)$ and $y''(x)$ respectively on $\{c_{n+j}\}$. Moreover equations (1.3),(1.4) and (1.5) can be expressed in the matrix-vector form as:

$$\mathrm{D}\alpha = y \tag{1.6}$$

Where the p-square matrix D, the p-vector α and y are defined as follows:

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & . & . & x_n^{p-1} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & . & . & x_{n+1}^{p-1} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & x_{n+r-1} & x_{n+r-1}^2 & x_{n+r-1}^3 & x_{n+r-1}^4 & . & . & x_{n+r-1}^{p-1} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & . & . & D'x_n^{p-2} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 1 & 2x_{n+r-1} & 3x_{n+r-1}^2 & 4x_{n+r-1}^3 & . & . & D'x_{n+r-1}^{p-2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & . & . & D''x_n^{p-3} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 2 & 6x_{n+r-1} & 12x_{n+r-1}^2 & . & . & D''x_{n+r-1}^{p-3} \end{pmatrix} \tag{1.7}$$

$$\alpha = (\alpha_0, \alpha_1, \alpha_2 \ldots \alpha_{p-1})^T . \, y = (y_n \ldots \ldots . \, y_{n+r-1}, \bar{y}'_n \ \ldots \ldots . \, \bar{y}'_{n+r-1}, \overline{y''}_n \ \ldots \ldots . \, \overline{y''}_{n+r-1})^T. \tag{1.8}$$

Where D' and D'' represent first and second derivatives respectively.

3 Derivation of the Two Step SDHBBDF Block Method

Consider the interpolation and collocation method defined for the step $|x_n, x_{n+1}|$ by

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + h\beta_{k-1}(x) f_{n+k-1} + h\beta_{k+1}(x) f_{n+k+1} + h^2\gamma_{k-1}(x) g_{n+k-1} + h^2\gamma_{k+1}(x) g_{n+k+1} \tag{1.9}$$

For $k = 2$ The matrix D, in (1.7) becomes

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^3 & 6x_{n+1}^5 \\ 0 & 1 & 2x_{n+3/2} & 3x_{n+3/2}^2 & 4x_{n+3/2}^3 & 5x_{n+3/2}^3 & 6x_{n+3/2}^5 \\ 0 & 1 & 6x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^3 & 6x_{n+2}^5 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 15x_{n+1}^3 & 30x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 15x_{n+2}^3 & 30x_{n+2}^3 \end{pmatrix} \tag{2.0}$$

Using Maple software, the inverse of the matrix in (2.6) is obtained and this yields the elements of the matrix D. The element of the matrix D substituted into (2.8) yields the continuous formulation of the method as:

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_{n+\frac{3}{2}}(x)f_{n+\frac{3}{2}} + \beta_2(x)f_{n+2}\right] + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+1} + \gamma_2(x)g_{n+2}] \tag{2.1}$$

Evaluating (2.7) at x_n, $x_{n+3/2}$ and x_{n+2} yields the following discrete methods which constitute the new three step block method.

$$\begin{aligned} y_n - y_{n+1} &= \frac{h}{330}\left[733f_{n+1} - 1616f_{\frac{3}{2}} + 553f_{n+2}\right] + \frac{h^2}{660}[40g_n + 1019g_{n+1} - 219g_{n+2}] \\ y_{n+3/2} &= -\frac{1}{512}y_n + \frac{513}{512}y_{n+1} + \frac{3h}{1024}\left[99f_{n+1} + 80f_{n+\frac{3}{2}} - 9f_{n+2}\right] + \frac{9h^2}{2048}[7g_{n+1} + g_{n+2}] \\ y_{n+3/2} &= y_{n+1} + \frac{h}{30}\left[7f_{n+1} + 16f_{n+\frac{3}{2}} + 7f_{n+2}\right] + \frac{h^2}{60}[g_{n+1} - g_{n+2}] \end{aligned} \tag{2.2}$$

This new schemes is consistent and its orders are 6 with A-stable.

4 Derivation of the Three Step SDHBBDF Block Method

In this case, $k = 3$, and its continuous form expressed in the form of (2.8)

$$y(x) = \sum_{j=0}^{k}\alpha_j(x)y_{n+j} + h\beta_{k-1}(x)f_{n+k-1} + h\beta_{k+1}(x)f_{n+k+1} + h^2\gamma_{k-1}(x)g_{n+k-1} + h^2\gamma_{k+1}(x)g_{n+k+1} \tag{2.3}$$

Similarly, we generate the continuous formulation of the new three step method as:

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + h\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_{n+\frac{5}{2}}(x)f_{n+\frac{5}{2}} + \beta_3(x)f_{n+3}\right] + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+1} + \gamma_2(x)g_{n+2} + \gamma_3(x)g_{n+3}] \tag{2.4}$$

Evaluating (2.4) at the following points at $x_n, x_{n+1}, x_{n+5/3}$ *and* x_{n+3} yields the following discrete methods which constitute the new three step block method.

$$y_n - \frac{772528}{57999}y_{n+1} + \frac{714529}{57999}y_{n+2} = -\frac{2h}{57999}\left[488303f_{n+2} - 1197568f_{\frac{5}{2}} + 381000f_{n+3}\right] + \frac{2h^2}{521991}[25595g_n - 3851676g_{n+2} + 649828g_{n+3}]$$

$$y_n + \frac{128752}{5659} y_{n+1} - \frac{134411}{5659} y_{n+2} = \frac{6h}{5659}\left[22959 f_{n+2} - 66304 f_{\frac{5}{2}} + 20000 f_{n+3}\right] + \frac{2h^2}{16977}[25595 g_{n+1} + 239633 g_{n+2} - 33319 g_{n+3}]$$

$$y_{n+5/3} = \frac{243}{2620928} y_n - \frac{365}{81904} y_{n+1} + \frac{2632365}{2620928} y_{n+2} + \frac{15h}{1310464}\left[26037 f_{n+2} + 19072 f_{\frac{5}{2}} - 1800 f_{n+3}\right] + \frac{45h^2}{1310464}[1027 g_{n+2} + 94 g_{n+3}]$$

$$y_{n+3} = -\frac{1}{25595} y_n + \frac{27}{25595} y_{n+1} + \frac{25569}{25595} y_{n+2} + \frac{6h}{25595}\left[981 f_{n+2} + 2304 f_{\frac{5}{2}} - 985 f_{n+3}\right] + \frac{18h^2}{25595}[21 g_{n+2} - 23 g_{n+3}] \quad (2.5)$$

This new schemes is consistent and its orders are 7 with A-stable have the following error constants as can be shown in Fig. 2 and in Table 2.

5 Analysis of the New Methods

In this research, we consider the analysis of the newly constructed methods. Their convergence is determined and their regions of absolute stability are plotted.

6 Convergence

The convergence of the new block methods is determined using the approach by Fatunla [8] and Chollom et al. [9] for linear multistep methods, where the block methods are represented in single block, r point multistep method of the form

$$A^{(0)} y_{m+1} = \sum_{i=1}^{k} A^{(i)} y_{m+1} + h \sum_{i=0}^{k} B^{(i)} f_{m+1} \quad (2.6)$$

Where h is a fixed mesh size within a block, $A^i, B^i, i = 0,1,2, \ldots . k$ are $r \times r$ identity while $y_m, y_{m-1} and\ y_{m+1}$ are vectors of numerical estimates.

Definition: A numerical method is said to be A-stable if the whole of the left-half plane $\{Z: Re(Z) \leq 0\}$ is contained in the region. $\{Z: Re(Z) \leq 1\}$ Where $R(Z)$ is called the stability polynomial of the method (Lambert [10]).

The block method (2.2) expressed in the form of (2.6) gives

$$\begin{pmatrix} -1 & 0 & 0 \\ -\frac{513}{512} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{512} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h\left[\begin{pmatrix} \frac{733}{330} & -\frac{808}{165} & \frac{553}{330} \\ \frac{297}{1024} & \frac{15}{64} & -\frac{27}{1024} \\ \frac{7}{30} & \frac{8}{15} & \frac{7}{30} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}\right]$$

$$+h^2\left[\begin{pmatrix} \frac{1019}{660} & 0 & -\frac{73}{220} \\ \frac{63}{2048} & 0 & \frac{9}{2048} \\ \frac{1}{60} & 0 & -\frac{1}{60} \end{pmatrix} \begin{pmatrix} g_{n+1} \\ g_{n+3/2} \\ g_{n+2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{2}{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{n-2} \\ g_{n-1} \\ g_n \end{pmatrix}\right] \quad (2.7)$$

Where $A^{(0)} = \begin{pmatrix} -1 & 0 & 0 \\ -\frac{513}{512} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $A^{(1)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{512} \\ 0 & 0 & 0 \end{pmatrix}$, $B^{(0)} = \begin{pmatrix} \frac{733}{330} & -\frac{808}{165} & \frac{553}{330} \\ \frac{297}{1024} & \frac{15}{64} & -\frac{27}{1024} \\ \frac{7}{30} & \frac{8}{15} & \frac{7}{30} \end{pmatrix}$, $B^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C^{(0)} = \begin{pmatrix} \frac{1019}{660} & 0 & -\frac{73}{220} \\ \frac{63}{2048} & 0 & \frac{9}{2048} \\ \frac{1}{60} & 0 & -\frac{1}{60} \end{pmatrix}$

, $C^{(1)} = \begin{pmatrix} 0 & 0 & \frac{2}{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Substituting $A^{(0)}$ and $A^{(1)}$ into (2.6) gives the characteristic polynomial of the block method $\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)})$

$$= det\left[\lambda \begin{pmatrix} -1 & 0 & 0 \\ -\frac{513}{512} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{512} \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= det\begin{bmatrix} -\lambda & 0 & 1 \\ -\frac{513}{512} & \lambda & \frac{1}{512} \\ -\lambda & 0 & \lambda \end{bmatrix}$$

$$= \lambda^2(\lambda - 1) = 0$$

Therefore, $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$. The block method (2.6) by definition is A-stable and by Henrici [11], the block method is convergent.

Similarly, the block methods (2.5) expressed in the form of (2.6) gives

$$\begin{pmatrix} -\frac{772528}{57999} & \frac{714529}{57999} & 0 & 0 \\ \frac{128752}{5659} & -\frac{134411}{5659} & 0 & 0 \\ \frac{365}{81904} & -\frac{2632365}{2620928} & 1 & 0 \\ -\frac{27}{25595} & -\frac{25569}{25595} & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+5/2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \frac{243}{2620928} \\ 0 & 0 & 0 & -\frac{1}{25595} \end{pmatrix} \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$
$$+ h\left[\begin{pmatrix} 0 & -\frac{976606}{57999} & \frac{2395136}{57999} & -\frac{254000}{19333} \\ 0 & \frac{137754}{5659} & -\frac{397824}{5659} & \frac{120000}{5659} \\ 0 & \frac{390555}{1310464} & \frac{2235}{10238} & -\frac{3375}{163808} \\ 0 & \frac{5886}{25595} & \frac{13824}{25595} & \frac{1182}{5119} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+5/2} \\ f_{n+3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}\right]$$

$$+h^2\left[\begin{pmatrix} 0 & -\frac{855928}{57999} & 0 & \frac{1299656}{521991} \\ \frac{51190}{16977} & \frac{479266}{16977} & 0 & -\frac{66638}{16977} \\ 0 & \frac{46215}{1310464} & 0 & \frac{2115}{655232} \\ 0 & \frac{378}{25595} & 0 & -\frac{414}{25595} \end{pmatrix}\begin{pmatrix} g_{n+1} \\ g_{n+2} \\ g_{n+5/2} \\ g_{n+3} \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & \frac{51190}{521991} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{n-3} \\ g_{n-2} \\ g_{n-1} \\ g_{n} \end{pmatrix}\right] \tag{2.8}$$

Where $A^{(0)} = \begin{pmatrix} -\frac{772528}{57999} & \frac{714529}{57999} & 0 & 0 \\ \frac{128752}{5659} & -\frac{134411}{5659} & 0 & 0 \\ \frac{365}{81904} & -\frac{2632365}{2620928} & 1 & 0 \\ -\frac{27}{25595} & -\frac{25569}{25595} & 0 & 1 \end{pmatrix}, A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \frac{243}{2620928} \\ 0 & 0 & 0 & -\frac{1}{25595} \end{pmatrix}$ (2.9)

$$B^{(0)} = \begin{pmatrix} 0 & -\frac{976606}{57999} & \frac{2395136}{57999} & -\frac{254000}{19333} \\ 0 & \frac{137754}{5659} & -\frac{397824}{5659} & \frac{120000}{5659} \\ 0 & \frac{390555}{1310464} & \frac{2235}{10238} & -\frac{3375}{163808} \\ 0 & \frac{5886}{25595} & \frac{13824}{25595} & \frac{1182}{5119} \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0}$$

$$C^{(0)} = \begin{pmatrix} 0 & -\frac{855928}{57999} & 0 & \frac{1299656}{521991} \\ \frac{51190}{16977} & \frac{479266}{16977} & 0 & -\frac{66638}{16977} \\ 0 & \frac{46215}{1310464} & 0 & \frac{2115}{655232} \\ 0 & \frac{378}{25595} & 0 & -\frac{414}{25595} \end{pmatrix}, C^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \frac{51190}{521991} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Substituting $A^{(0)}$ and $A^{(1)}$ into (2.6) gives the characteristic polynomial of the block method $\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)})$

$$= det\left[\lambda\begin{pmatrix} -\frac{772528}{57999} & \frac{714529}{57999} & 0 & 0 \\ \frac{128752}{5659} & -\frac{134411}{5659} & 0 & 0 \\ \frac{365}{81904} & -\frac{2632365}{2620928} & 1 & 0 \\ -\frac{27}{25595} & -\frac{25569}{25595} & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \frac{243}{2620928} \\ 0 & 0 & 0 & -\frac{1}{25595} \end{pmatrix}\right] \tag{3.1}$$

$$= det\begin{bmatrix} \frac{-772528}{57999}\lambda & \frac{714529}{57999}\lambda & 0 & 1 \\ \frac{128752}{5659}\lambda & \frac{-134411}{5659}\lambda & 0 & 1 \\ \frac{365}{81904}\lambda & \frac{-2632365}{2620928}\lambda & \lambda & \frac{-243}{2620928} \\ \frac{-27}{25595}\lambda & \frac{-25569}{25595}\lambda & 0 & \lambda - \frac{1}{25595} \end{bmatrix} \tag{3.2}$$

$$= \lambda^4 - \lambda^3 = 0$$

Therefore, $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 0$. The block method (2.6) by definition is A-stable and by Henrici [11], the block method is convergent.

The hybrid block methods which are obtained in a block form with the of maple software have the following order and error constant for each case.

K=2 **SDHBBDF WITH ONE OFF-SET POINT.**

The method K=2 is of order 6 as a block and has error constant

$$C_7 = \left(\frac{5659}{6652800}, \quad \frac{-9}{2293760}, \quad \frac{1}{604800}\right)^T$$

K=3 **SDHBBDF WITH ONE OFF-SET POINT.**

The method K=3 is of order 7 as a block and has error constant

$$C_8 = \left(\frac{5659}{6652800}, \quad \frac{-9}{2293760}, \quad \frac{1}{604800}, \quad \frac{1}{604800}\right)^T$$

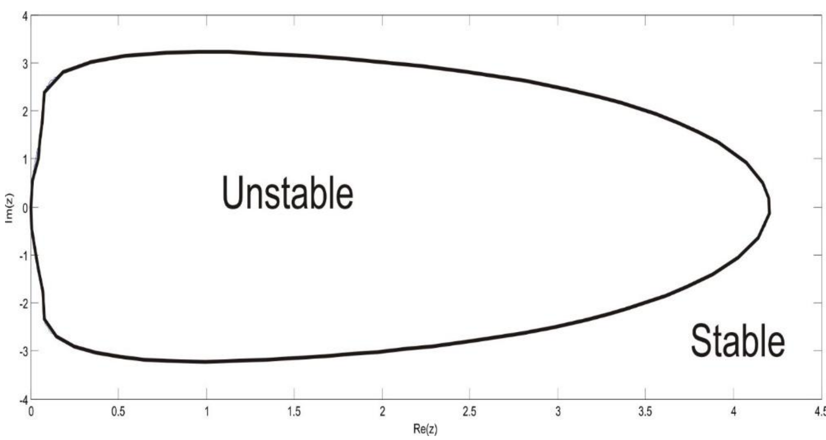

Fig. 1. Absolute stability regions of the new two step discrete methods

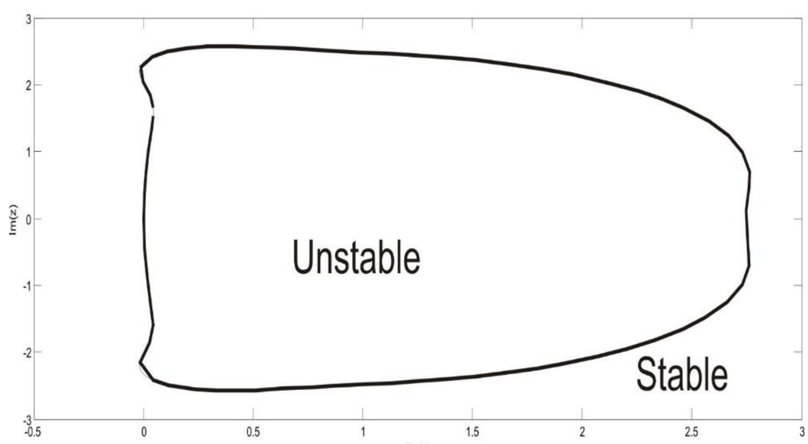

Fig. 2. Absolute stability regions of the new three step discrete methods

8 Numerical Implementation

Problem 1: Consider the stiffly linear problem of the system

$$y_1' = 998y_1 + 1998y_2 \qquad y_1(0) = 1$$
$$y_2' = -999y_1 - 1999y_2 \qquad y_2(0) = 0$$
$$h = 0.1 \qquad x \in [0,1]$$

Exact solution $y_1(x) = 2e^{-x} - e^{-1000x}$
$y_2(x) = -e^{-x} - e^{1000x}$

Table 1. Absolute and relative errors of numerical solutions of problem 1 within the interval $0 \le x \le 1$

	HBSDBDF		$k = 2$	
	Absolute errors	**Relative errors**	**Absolute errors**	**Relative errors**
x	y_1	y_1	y_2	y_2
0.1	2.432×10^{-2}	1.344×10^{-2}	2.432×10^{-2}	2.688×10^{-2}
0.2	3.871×10^{-2}	2.139×10^{-2}	3.811×10^{-2}	4.728×10^{-2}
0.3	9.306×10^{-4}	6.281×10^{-4}	9.326×10^{-4}	1.259×10^{-3}
0.4	1.509×10^{-3}	1.126×10^{-3}	1.507×10^{-3}	2.248×10^{-3}
0.5	2.320×10^{-5}	1.912×10^{-5}	2.203×10^{-5}	3.632×10^{-5}
0.6	6.993×10^{-5}	6.371×10^{-5}	7.135×10^{-5}	1.300×10^{-4}
0.7	2.147×10^{-5}	2.162×10^{-5}	1.223×10^{-5}	2.463×10^{-5}
0.8	2.339×10^{-5}	2.603×10^{-5}	1.455×10^{-5}	3.238×10^{-5}
0.9	2.169×10^{-5}	2.667×10^{-5}	1.604×10^{-5}	3.945×10^{-5}
1.0	1.971×10^{-5}	2.679×10^{-5}	1.483×10^{-5}	4.031×10^{-5}

Table 2. Absolute and relative errors of numerical solutions of problem 1 within the interval $0 \le x \le 1$

x	HBSDBDF		$k = 3$	
	Absolute errors	**Relative errors**	**Absolute errors**	**Relative errors**
	y_1	y_1	y_2	y_2
0.1	5.817×10^{-2}	3.214×10^{-2}	5.825×10^{-2}	6.348×10^{-2}
0.2	4.023×10^{-3}	2.457×10^{-3}	3.950×10^{-3}	4.728×10^{-2}
0.3	9.169×10^{-3}	6.188×10^{-3}	2.162×10^{-3}	2.918×10^{-3}
0.4	7.126×10^{-4}	5.315×10^{-4}	6.275×10^{-4}	9.361×10^{-4}
0.5	1.204×10^{-4}	9.925×10^{-5}	4.215×10^{-5}	6.949×10^{-5}
0.6	2.281×10^{-4}	2.078×10^{-4}	1.573×10^{-4}	2.866×10^{-4}
0.7	1.604×10^{-4}	1.615×10^{-4}	7.768×10^{-5}	1.564×10^{-4}
0.8	1.519×10^{-4}	1.690×10^{-4}	7.613×10^{-5}	1.694×10^{-4}
0.9	1.364×10^{-4}	1.677×10^{-4}	6.782×10^{-5}	1.668×10^{-4}
1.0	1.519×10^{-4}	2.064×10^{-4}	7.595×10^{-5}	2.064×10^{-4}

Problem 2: Consider A two-dimensional SODEs which have been solved by (Ibrahim et al. [12]).

$$y_1' = 198y_1 + 199y_2 \qquad y_1(0) = 1$$
$$y_2' = -398y_1 - 399y_2 \qquad y_2(0) = -1$$
$$h = 0.1 \qquad x \in [0,1]$$

Exact solution $y_1(x) = e^{-x}$
$y_2(x) = -e^{-x}$

Table 3. Absolute and relative errors of numerical solutions of problem 2 within the interval $0 \le x \le 1$

x	HBSDBDF		$k = 2$	
	Absolute errors	**Relative errors**	**Absolute errors**	**Relative errors**
	y_1	y_1	y_2	y_2
0.1	3.605×10^{-7}	3.984×10^{-7}	3.598×10^{-7}	3.976×10^{-7}
0.2	3.211×10^{-7}	3.549×10^{-7}	3.204×10^{-7}	3.913×10^{-7}
0.3	6.278×10^{-7}	8.474×10^{-7}	6.273×10^{-7}	8.468×10^{-7}
0.4	5.650×10^{-7}	7.627×10^{-7}	5.650×10^{-7}	8.429×10^{-7}
0.5	6.685×10^{-7}	1.102×10^{-6}	6.680×10^{-7}	1.101×10^{-6}
0.6	6.025×10^{-7}	1.098×10^{-6}	6.019×10^{-7}	1.097×10^{-6}
0.7	5.918×10^{-7}	1.192×10^{-6}	5.916×10^{-7}	1.191×10^{-6}
0.8	5.368×10^{-7}	1.195×10^{-6}	5.365×10^{-7}	1.194×10^{-6}
0.9	7.380×10^{-7}	1.815×10^{-6}	7.376×10^{-7}	1.814×10^{-6}
1.0	6.703×10^{-7}	1.822×10^{-6}	6.700×10^{-7}	1.821×10^{-6}

Table 4. Absolute and Relative errors of numerical solutions of problem 2 within the interval $0 \le x \le 1$

x	HBSDBDF		$k = 3$	
	Absolute errors	**Relative errors**	**Absolute errors**	**Relative errors**
	y_1	y_1	y_2	y_2
0.1	2.600×10^{-6}	2.873×10^{-6}	2.595×10^{-6}	2.868×10^{-6}
0.2	2.419×10^{-6}	2.955×10^{-6}	2.419×10^{-6}	2.955×10^{-6}
0.3	2.177×10^{-6}	2.939×10^{-6}	2.177×10^{-6}	2.939×10^{-6}
0.4	3.898×10^{-6}	5.815×10^{-6}	3.894×10^{-6}	5.809×10^{-6}
0.5	3.577×10^{-6}	5.897×10^{-6}	3.577×10^{-6}	5.897×10^{-6}
0.6	3.227×10^{-6}	5.880×10^{-6}	3.227×10^{-6}	5.880×10^{-6}
0.7	4.349×10^{-6}	8.758×10^{-6}	4.346×10^{-6}	8.752×10^{-6}
0.8	3.971×10^{-6}	8.838×10^{-6}	3.972×10^{-6}	8.840×10^{-6}
0.9	3.587×10^{-6}	8.823×10^{-6}	3.587×10^{-6}	8.823×10^{-6}
1.0	4.305×10^{-6}	1.170×10^{-5}	4.303×10^{-6}	1.170×10^{-6}

9 Conclusion

The new SDHBBDF method with one off-set point at collocation of two and three steps with order 6 and 7 for the solution of systems of linear stiff ordinary differential equations. The methods are A-stable and are powerful for handling stiff problems. The numerical result showed that SDHBBDF methods are efficient and highly competitive compared to existing methods solved stiff problems.

Competing Interests

Authors have declared that no competing interests exist.

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