Scientific Research

Transformation of Nonlinear Surface Gravity Waves under Shallow-Water Conditions

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Abstract

This article describes transformation of nonlinear surface gravity waves under shallow-water conditions with the aid of the suggested semigraphical method. There are given profiles of surface gravity waves on the crests steepening stages, their leading edges steepening. There are discussed the spectral component influence on the transformation of surface wave profile.

1. Introduction

Surface gravity waves under shallow-water conditions were of great interest for many researchers many years ago. In spite of their magnetic view, they are rather hard to describe. The research problems of surface waves in the near-shore region within shallow-water model are discussed in [1,2]. [3] is concerned with nonlinear waves in strongly dispersive media. Transformation of surface waves in the near-shore region is discussed in [4]. The nonlinear dynamics of surface gravity waves in long-wave approximation is described in [5,6]. [6] analysed the process of surface wave deformation under shallow-water conditions, estimated spectral amplitude of nonlinear wave with its steepness.

In works [7,8] give mathematic simulation of gravity waves in approximation of shallow water. [8] is concerned with numerical simulation of water waves within nonlinear-dispersive model of shallow water taking account of bed topography. Experimental data on influence of the dispersion effects on the propagation of nonlinear surface waves in inshore area are given in [9].

2. Statement of the Problem

In work [10] discussed the problem of nonlinear surface waves propagation under shallow-water conditions. Shallow-water equation taking account of quadratic nonlinearity was solved by the method of successive approximations.

Either the shallow-water equation or the method of successive approximations is the basis of nonlinear wave research. In the middle of the 19th century G. Airy described tide waves and G. Stokes described waves of finite amplitude with this equation and this method [11]. However, the generating process of higher harmonics, resulting in wave crest sharpening, did not always meet energy conservation law. When higher harmonics are growing, the decrease of principal wave energy is ignored. The exact solution of shallow-water equation without dispersion and damping appears to be Riemannian invariants, based on different propagation velocity of wave crests and troughs [12,13]. In addition, shallow-water equation ignores dispersion effect because of its weakening on shallow water.

Surface gravity waves on shallow water in approximation of small nonlinearity and small dispersion are described by well known Cortvega and De Vries equation [11,12]. In the equation small nonlinearity, resulting in wave steepening, and small dispersion, resulting in wave diffusion, compensate each other. Thereat, stationary nonlinear wave (named as a cnoidal wave) occurs, which propagates without form change at a constant velocity. However, form stability of cnoidal waves doesn't allow retracing the dynamics of nonlinear surface gravity waves with the propagation on the shallow water.

In the development of earlier studies this work describes the transformation process of surface wave profile with the propagation on the shallow water. There was an attempt to describe not only the initial wave crest sharp-

ening but further steepening of its leading edge. One uses rather understandable and vivid semigraphical method based on energy conservation law given in [10].

3. Research Method

Method of successive approximations is the straightest way of the solution of nonlinear Equations [14]. Here, the small parameter expansion $\varepsilon (\varepsilon \ll 1)$ is used, restricting to the first two terms under the restriction $u^{(n+1)} \ll u^{(n)}$.

The equation taking account of quadratic nonlinearity takes the form [12,13]:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = -\varepsilon u \frac{\partial u}{\partial x} \tag{1}$$

where $u = \vartheta_x$ — the horizontal component of medium particle velocity, nonlinearity — $\varepsilon = \dfrac{\vartheta_x}{c} = \dfrac{a}{H}$, the propagation velocity of gravity waves $c = \sqrt{gH}$, a — the amplitude of the vertical shift of the free surface, H — the fluid depth.

With the propagation of the initial harmonic wave on shallow water as a consequence of nonlinear effects, the second harmonic appears. With the growth of the second harmonic its interaction with first harmonic will get stronger. This interaction will lead to the exciting of the third, fourth, and so on, harmonics.

In the first approximation, solution of Equation (1) consists of primary waves and takes the form [10]:

$$u_{prim.}^{(1)}(x,t) = \varepsilon u^{(1)}(x,t) + \varepsilon^2 u^{(2)}(x,t) = \\ = \varepsilon U_0 \exp[ikx] - ikt\varepsilon^2 U_0^2 \exp[i2kx] + c.c. \tag{2}$$

In the second approximation the non-homogeneous linear equation is solved

$$\frac{\partial u_{sec.}^{(2)}}{\partial t} + c \frac{\partial u_{sec.}^{(2)}}{\partial x} = -\frac{\partial u_{prim.}^{(1)2}}{2\partial x}. \tag{3}$$

After squaring and solution differentiation in the first approximation (2) we obtain the expression in the second approximation:

$$u_{sec.}^{(2)}(x,t) = \left[\left(t\varepsilon^2 U_0^2 / 2 \right) \cdot ik_2 \exp[ik_2 x] + \right. \\ + \left(-k^2 t^3 \varepsilon^4 U_0^4 / 2 \right) \cdot ik_4 \exp[ik_4 x] + \\ + \left(-ikt^2 \varepsilon^3 U_0^3 \right) \cdot ik_3 \exp[ik_3 x] + \\ \left. + \left(-ikt^2 \varepsilon^3 U_0^3 \right) \cdot ik_1 \exp[ik_1 x] \right] + (c.c.) \tag{4}$$

where $k_n = nk$, $n = 1, 2, 3, 4$ — harmonic number.

As a consequence of interaction between principal wave k and second $2k$ secondary waves with double values $2k$, $4k$, and complex waves k and $3k$ will occur. With

time, higher harmonics in the spectrum lead to the distortion of wave profile.

The solution of shallow-water Equation (1) in two approximations for horizontal velocity of medium particles will take the form

$$u_{sec.}(x,t) = \varepsilon u_{prim.}^{(1)}(x,t) + \varepsilon^2 u_{sec.}^{(2)}(x,t) = \\ = \left[\varepsilon^2 U_0 \exp[ikx] - ikt\varepsilon^3 U_0^2 \exp[2ikx] \right] + \\ + \left[kt^2 \varepsilon^5 U_0^3 \cdot k_1 \exp[ik_1 x] + \left(t\varepsilon^4 U_0^2 / 2 \right) \cdot ik_2 \exp[ik_2 x] + \right. \\ + kt^2 \varepsilon^5 U_0^3 \cdot k_3 \exp[ik_3 x] + \\ \left. + \left(-k^2 t^3 \varepsilon^6 U_0^4 / 2 \right) \cdot ik_4 \exp[ik_4 x] + \right] + (c.c.) \tag{5}$$

4. Simulation within the Gulf

To check the results obtained we will use hydrologic conditions of the Gulf of Taganrog of the Azov Sea. The mean depth of the gulf does not exceed 5 m, therefore, the shallow-water conditions will be satisfied with gravity waves with their lengths are not longer than 30 m, the gulf bed is flat, surface tension is absent, the effect of wind is not taken into account.

Before calculation one should mark some physical features of existent wave processes. In formula (5) all harmonic amplitudes are always growing because of secular terms. However, according to the formula, the first harmonic amplitude remains constant at that. Though, higher harmonics are fared energetically from the first harmonic.

It follows that the primary waves amplitude in formula (5) $u_{prim.}^{(1)}(x,t)$ should be getting decreased. That is why to research profile dispersions of a gravity wave it is necessary to consider these physical features of wave processes otherwise it will break the energy conservation law.

In the absence of dissipation, energies of primary and secondary waves should satisfy должны отвечать the energy conservation law

$$E_1^{(1)}(x) + E_2^{(1)}(x) = E_1^{(2)}(x) + E_2^{(2)}(x) + E_3^{(2)}(x) + E_4^{(2)}(x) \tag{6}$$

where $E_n^{(1)}(x)$, $E_n^{(2)}(x)$ — energies of primary and secondary waves.

Let us follow variations the profile change of the gravity wave after coming into the gulf with the initial parameters: frequency $f = 0.09$ Hz; length $\lambda = 77.8$ m; initial steepness — $2a/\lambda = 0.014$; $a = 0.537$ m; $kH = 0.4$. According to **Figure 1(a)** for this wave instability appears at the distance: $x_{inst.}^{(2)} \approx 10$ km (i.e. in $t_{inst.}^{(2)} \approx 24$ min.). Though our model is correct only within $x \approx 6$ km, i.e. till, while the primary limitation $u^{(n+1)} \ll u^{(n)}$ is fulfilled,
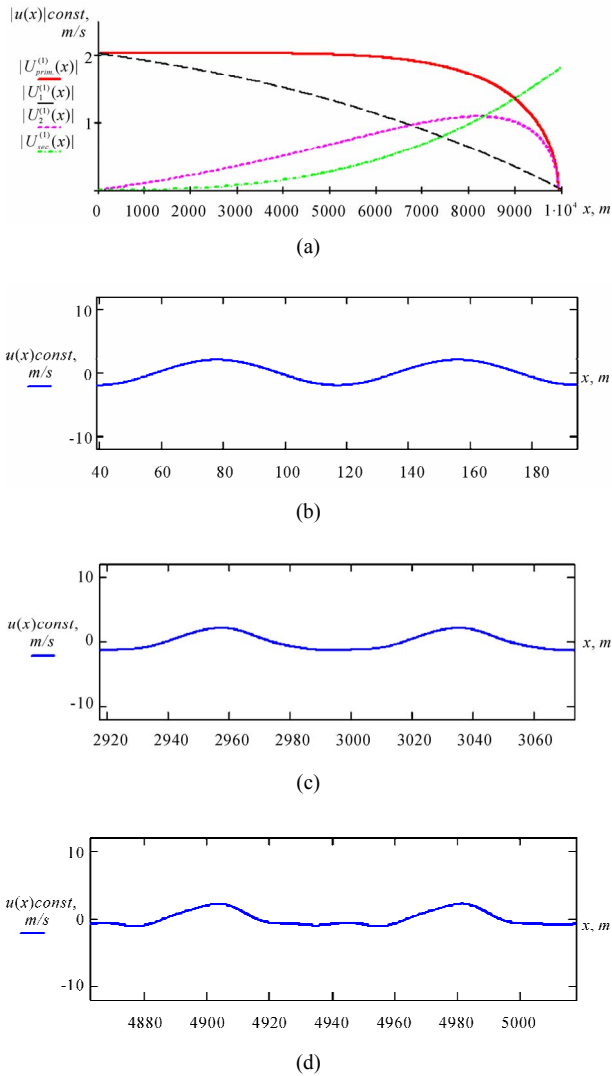
(a)

(b)

(c)

(d)

Figure 1. Primary and secondary wave velocities (a) and profiles (b, c, d) of the surface gravity wave: f = 0.09 Hz; λ = 77.8 m; wave parameter – kH = 0.4; initial steepness – $2a/\lambda$ = 0.014; nonlinear parameter – ε = 0.107.

when the value for the second approximation is by an order less than the value for the initial approximation.

Because of **Figure 1(a)** analysis, it is possible point out that the amplitude of primary waves and first harmonic falls with the growth of secondary waves summary amplitude. The amplitude of the second harmonic grows slowly, and to the moment of instability it falls down as there is a complete energy transfer from primary waves (principal wave and the second harmonic) to secondary waves.

With the growth of initial steepness the instability distance decreases, with the growth of the wave amplitude (with the constant steepness) this distance lengthens, as nonlinear parameter value decreases.

Let us follow the profile change of the initial gravity

wave within distance $x \approx 6$ km. **Figures 1(b)**, **1(c)** and **1(d)** describe dependences of horizontal velocity change $u(x,t)$ on the distance run taking account of the formula (5) (expanded vertical scale). The wave with an initially cosinusoidal profile is gradually distorted in a run distance time, the crests steepen **Figure 1(c)**, and the wave troughs flatten. Thereafter there is a steepening of the wave leading edge **Figure 1(d)**. This is the result of the increasing influence of high frequency harmonics. The increasing of particle velocity on the crest leads to the further steepening of the wave leading edge. The breaking of such a wave in shallow water happens as a plunging or spilling breaker [15,16]. The breaking cause in our case is naturally liquid depth.

An additional point to emphasize is that the shallow-water condition could not be observed for the higher harmonics appear. It could lead to the propagation velocity dispersion that could cause braking of nonlinear process. But approaching the coast as the depth lessens, the shallow-water condition will be met better, therefore the wave breaks.

To analyse the secondary waves influence **Figure 2** gives graphs showing the growth of velocity amplitude of the secondary waves because of the distance. It clearly enough shows the relation between harmonics. Within the secondary waves, velocity amplitude of the fourth harmonic increases faster (because of the greatest degree of a secular term in the formula (5)), then there comes the third one, the first one and the second one. This conformity is observed for the gravity waves with other initial data as well.

5. Analysis and the Result Comparison

Analysing the constructed profiles of the surface gravity waves it is necessary to note their following features. From the field studies it is known that the profile of real surface waves is asymmetrical: the crests are steep and short, sharpened, and troughs are flattened and broad [11]. Undulations are asymmetrical as well: particles velocity below the crests is lower than below the troughs. Theses peculiarities are explained by Stokes theory and the ap-
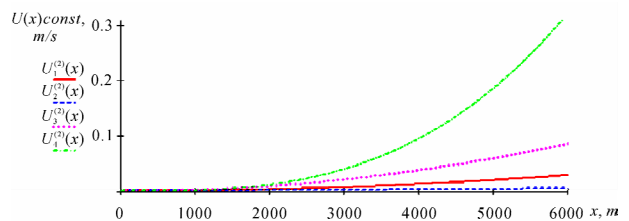

Figure 2. The increase of velocity amplitudes of secondary waves from the distance with: f = 0.09 Hz; λ = 77.8 m; kH = 0.4; $2a/\lambda$ = 0.014; ε = 0.107.

pearance of principal wave high frequency harmonics. For our study the surface waves obtain such profiles on the initial propagation phases, it is shown in **Figure 1(b)**. On this stage higher harmonics generation leads to the crests sharpening, at that the crests keep symmetrical relating to a vertical axis.

Beside the asymmetry type described above, there is another asymmetry type of the real surface wave profile. It is expressed in such a way that the advancing side of the transformed wave gets steeper than the wave tail [17-19]. This difference in steepness ratio increases with the approaching to the coast. This asymmetry type of the real surface wave profile is not described by Stokes theory as well as the major part of other theories. Though to describe the process of wave climb from the open sea to the coast G. Whitham offered the equation with an integral element based on Cortvega and De Vries equation. On the wave breaking stage the operation of an integral element becomes modest and the solutions of Whitham equation are analogue to the solutions of simplest equations with nonlinear nature (analogue to the Equation (1)), falling and breaking for a final time [12,20].

For our study the surface wave profiles in the third propagation area just as obtain the second asymmetry kind, **Figure 1(d)** At that, steepening becomes more and more intense with the growth of the wave length, *i.e.* with the increase of nonlinearity factor.

One of the explanations of steepening of crest advancing side is the following: higher harmonics gradually displace in phase relating to the principal wave [16,21,22]. When phase displacement approaches the value of *π/2*, the advancing side becomes almost vertical; as a result the wave could fall. In the obtained resultant expression for particle horizontal velocity (5) even harmonics amplitudes are imaginary. With the increase of secondary waves the imaginary is felt stronger than in the beginning as it leads to the phase displacement relative to the principal wave amplitude. It is necessary to note that there is no phase difference between the principal wave and higher harmonics in Stokes theory, *i.e.* it couldn't describe these processes.

To show the influence of even harmonics, especially the fourth one, the surface gravity wave profile, **Figure 3** gives graphs showing the horizontal velocity *u(x,t)* of medium particles for the same propagation interval. The special influence of the fourth harmonic is connected with its rapid growth, it follows that the greatest energy-output ratio within secondary waves. **Figure 3(a)** gives the surface gravity wave profile, counted according to the formula (5), when the fourth harmonic in phase is behind the principal wave with *π/2*. **Figure 3(b)** gives the wave profile for the case, when the imaginary of the fourth harmonic amplitude is changed to be of real type, *i.e.* there is
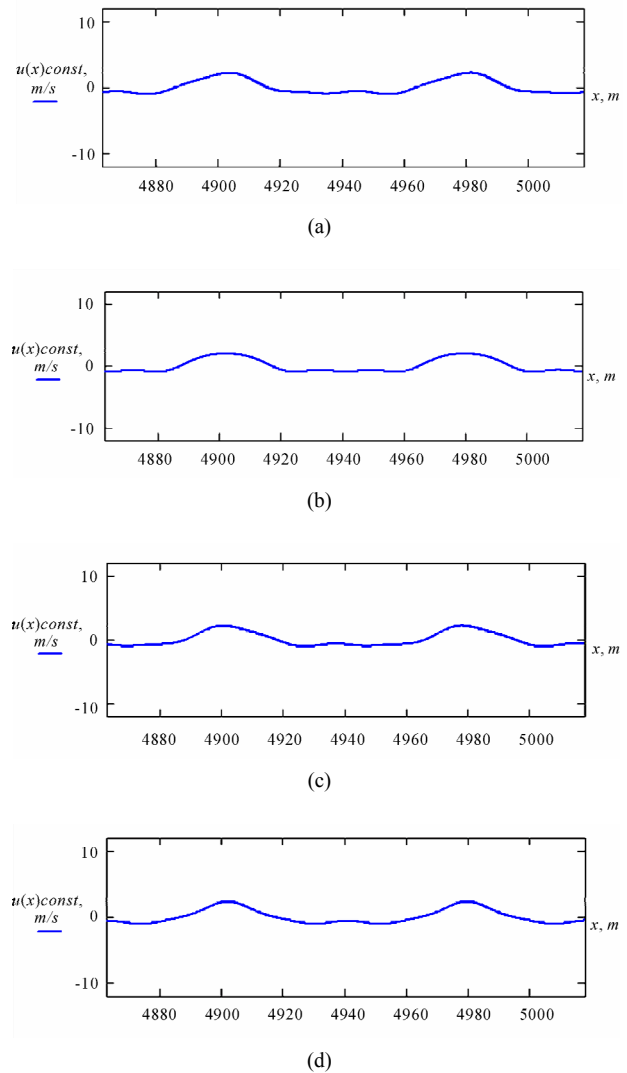

Figure 3. The fourth harmonic influence on the surface gravity wave profile: (a) is behind the principal wave with *π/2*; (b) subtracted in phase; (c) is ahead of the principal wave with *π/2*; (d) summarized in phase.

no the phase displacement between the harmonic and the principle wave. At that the fourth harmonic amplitude is negative, it follows that it is subtracted from the principle wave amplitude. It leads to the symmetric property of wave crest relating to vertical axis.

In formula (5) only the fourth harmonic amplitude is negative. The change of the negative sign to the positive one leads to the change of steepening of the wave crest front edge to the trailing edge. This case is given in **Figure 3(c)**, it reminds the profile of the reverse wave, the wave propagates backwards. In this case the fourth harmonic leads in phase the principle wave with *π/2*. If in formula (5) at the same time one changes the negative sign to the positive one, as well the imaginary of the fourth harmonic changes to be of real type, the wave

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profile takes the form as in **Figure 3(d)**.

In this case the fourth harmonic amplitude is summarized in phase with the principle wave amplitude, intensifying the wave crest. The given profile resembles Stokes waves of finite amplitude on the deep water (or trochoidal wave).

It is important to pay attention to the appearance of the local projection in the trough between the crests. This projection is also connected with the fourth harmonic, and it changes its location depending on the phase. Experimental observations with the trough between the principal wave crests in nearshore zone are given in [9,21,23].

6. Conclusions

As a consequence of the researches made, one could note that the suggested semigraphical method allows to describe not only the process of the initial sharpening of the surface waves during the propagation under shallow water conditions, and further steepening of their front edge. In our study we described the influence of the first four harmonics of the principle wave on profile transformation but the real wave spectrum is constant, and for more reasonable study it is necessary to record at least first eight harmonics. The record of higher harmonics will lead to more intense sharpening of wave crests and steepness growing of the wave leading edge.

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