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Application of Multiple Scale Method to a Discretized Financial PDE

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Abstract

This paper presents an application of two way variable expansion method (multiple scale) for the calculation of the periodic solutions, resulted from a Hopf bifurcation of a discretized generic PDE in finance to a first order time-delay system arising from laser dynamics and a single inertial neural model with time delay. The two way variable expansion methods involve easy computation only, and yield estimation to the oscillatory movement of the price of stock with high accuracy.

Keywords: Multiple scale method; discretized financial PDE; hopf bifurcation; stock price.

1 Introduction

Over the past decade, rapid advances has been made to control and stabilize the transient behavior of some financial derivative [1]. From 2007 the Global financial Economy has been experiencing what is said to be worst financial crisis since the great depression in the 1930's. the current crisis is triggered by the short fall of liquidity in the United States, followed by collapsing of large financial institutions, bailout of banks, turn downs of international stock markets and credits, collapse of housing bubble, mortgage foreclosures, failure of key businesses, declines of wealth, increase of governmental debts due to substational commitments and many other factors. To those who have ever dealt with Black Scholes equations, the instabilities and oscillatory behavior modeled by Black Scholes equation are all too well known. What perhaps not so familiar is that the Black Scholes equations could be discretized into delay differential equation (DDE) which have the opposite effect in the financial market, namely that they could suppress oscillations and stabilize equilibria which would be unstable in the absence of delays. For instance, oscillatory behavior can often be connected to a Hopf bifurcation of an equilibrium solution under the variation of some parameter,

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and such local bifurcations share common qualities expressed in terms of the behavior on a low-dimensional center manifold. Hence, an analysis of stabilization near a generic Hopf bifurcation would yield general results applicable to any system near a Hopf instability and serve as a useful guide for understanding the behavior of a financial market systems under time delays.

Although the financial derivatives are governed by the celebrated parabolic partial differential Black-Scholes formula, but it is not clear how derivatives are controlled and stabilized. In this paper, analysis are made based on the discretization of Black-Scholes formula to a system of DDE's. It is found that such financial derivatives experience a drift which hardly can be brought to equilibrium state. In the case of ordinary differential equation (ODE's), a very popular method for obtaining transient behavior is the two variable expansion method (also known as multiple scales) [2,3,4] is proposed.

This paper is organized as follows. In section 2, we presented a scalar DDE and the properties it must satisfy for the existence of Hopf bifurcation. Illustrative example were also given. We review modeling of Black-Scholes and the partial differential equation which financial derivative have to satisfy in section 3. The generic PDE in finance were discretized in spatial dimension in section 4. Finally section 5, the method of multiple scale were presented, applied and analyzed.

2 Hopf Bifurcation of Time-delay Systems

Consider the following scalar DDE's with a parameter q (the delay or some other physical parameter)

$$x^{(n)}(t) = F(x(t), x^{'}(t), \dots, x^{n-1}(t), x(t-\tau), \\ x^{'}(t-\tau), \dots x^{n-1}(t-\tau), q)\ , x \in R \tag{1}$$

where F has at least up to fourth order continuous derivatives satisfying $F(0,0,\dots,0,q) \equiv 0$. Equation (1) is assumedto admit a Hopf bifurcation at $q = q_0$. The existence of the bifurcation can be characterized by the root location of the characteristic function $D(\lambda, q)$ of the linearized equation at $x = 0$ of (1) as follows:

- For a small $\varepsilon \coloneqq q - q_{0,}\ D(\lambda, q)$ has exactly one pair of simple complex roots $\lambda(\varepsilon) = \alpha(\varepsilon) \pm i\beta(\varepsilon)$ such that at $\varepsilon = 0$, one has $\alpha(0) = 0$, $w_0 = \beta(0) > 0$, and all the other characteristics roots have negative real parts.
- $\alpha^{'}(0) = R\frac{d\lambda}{d\varepsilon}(0) \neq 0$ (the transversality condition), where $R(z)$ stands for the real part of $z \epsilon \mathbb{C}$.

Due to the Hopf bifurcation Theory, the bifurcated nontrivial periodic solution has a period approximately${}^{2\pi}/_{\beta(\varepsilon)}$, and ${}^{2\pi}/_{\beta(\varepsilon)} \to {}^{2\pi}/_{w_0}$ as $\varepsilon \to 0$. Thus, in the vicinity of the Hopf bifurcation, namely for a sufficiently small $|\varepsilon|$, the stationary solution of (1) has a form

$$x(t) = r(\varepsilon t)\cos(w(\varepsilon)t + \theta) + 0(\varepsilon t)) \\ = r cos(w_o t + \theta) + 0(\varepsilon) \tag{2}$$

as done in applications of method of multiple scales, where

$r \coloneqq r(0), \theta \coloneqq \theta(0)$ for short. Therefore, it is expected that the time-delay system near the Hopf bifurcation behaves similar to the Black-Scholes differential equation involving a term $x^{''}(s)$. The key features of the Hopf bifurcation of (1) can be preserved if the right hand function F is approximated with the third or fifth order Taylor expansion, which is required in the computation of the averaged power function, defined in [5]. That is to say, the local dynamics near the Hopf bifurcation of (1) can be determined from the averaged power function [5].

Example1; let us study the following scalar DDE arising from laser physics [6].

$$\dot{x}(t) = -\left(\frac{\pi}{2} + \varepsilon\right) sinx(t-1) \tag{3}$$

where $|\ \varepsilon\ | \ll 1$ is a small parameter. Equation (3) undergoes a Hopf bifurcation at $\varepsilon = 0$, because the following conditions hold [7,1]:

1. For small $\varepsilon < 0$, the zero solution $x = 0$ of (3)is asymptotically stable.
2. At $\varepsilon = 0$, the characteristic function $p(\lambda) \coloneqq \lambda + (\frac{\pi}{2} + \varepsilon)e^{-\lambda}$ has a pair of complex conjugate roots $\lambda = \pm\, i\, {}^{\pi}/_{2}$, and the other roots of $p(\lambda)$ have negative real parts.
3. $R\left[\frac{d\lambda}{d\varepsilon}\right]_{\varepsilon=0} \neq 0$, where $R(z)$ stands for the complex conjugate of z.

The key features near the Hopf bifurcation can be determined from

$$\dot{x}(t) = -\left(\frac{\pi}{2} + \varepsilon\right) \times \left(x(t-1) - \frac{x^3(t-1)}{6} + \frac{x^5(t-1)}{120}\right) \tag{4}$$

because Hopf bifurcation is a local property of dynamical systems and also refers to the analysis or evaluation of market conditions based on two distinct scenarios.

3 Black-Scholes Financial Derivatives Overview

Considering the importance of financial derivatives, a crucial problem in finance is how to evaluate and price each financial derivative (option, futures and swap of a financial assets).

Black and Scholes [8] discovered the partial differential equation which financial derivatives (the underlying assets of which are stock) have to satisfy.

Merton's work [9] helps us to understand the Black-Scholes equation from the mathematical point of view.

Let t be time and s be the price of stock. Consider a derivative security whose price depends on s and t. The price is a function of s and t, so we call it $C(s,t)$ or just C. Then, our task is to find the equation which C satisfies. We assume that there is a risk-free bond B which earns a risk-free rate r.

That is, the following holds:

Bond (cash): A riskless B that evolves in accordance with the process

$$\mathrm{dB} = rBdt\,. \tag{5}$$

In addition, an underlying security which evolves in accordance with stock S that follows the geometric Brownian motion (Ito process):

$$\text{Stock: } dS = \mu Sdt + \sigma SdZ \tag{6}$$

here S is a Brownian motion, Z is a Wiener process, μ is constant parameter called the drift. It is a measure of the average rate of growth of the asset price. Meanwhile, σ is a deterministic function of time when σ is constant, (6) is the original Black-Scholes model of the movement of a security, S. In this formulation, μ is the mean return of S, and σ is the variance.

The quantity dZ is a random variable having a normal distribution with mean 0 and variance dt:

$$dZ \propto N(0, (\sqrt{dt})^2).$$

For each interval dt, dZ is a sample drawn from the distribution $N(0, (\sqrt{dt}\,)^2)$, this is multiplied by σ to produce the term σdZ. The value of the parameters μ and σ may be estimated from historical data.

Consider the derivative parabola

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S \frac{\partial C}{\partial S} dZ \tag{7}$$

from an option $C(s,t)$ written on the underlying security (by Ito's Lemma) evolves in accordance with the process. $C(s,t)$ is sufficiently smooth, namely, its second-order derivatives with respect to S and first-order derivative with respect to t are continuous in the domain. As it can be seen in Ito's Lemma, the price change is proportional to a coupled second order partial differential equation which depends on the random stochastic variable dZ, the deterministic function σ, and the drift parameter μ [10]. By comparing the portfolio G (a portfolio using B and S so that the portfolio behaves exactly the same with C) consisting of x shares of stock and y units of bond.

$$G = xS + yB \tag{8}$$

With the option $C(s,t)$, equation (5), (6), and (7), we derived

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0 \quad . \tag{9}$$

This partial differential equation (9) is celebrated Black Scholes equation. In this derivation (9), we replicated the derivative with a stock and a bond.

4 Discretising the Financial PDE Found in Finance

Consider the generic PDE for a contingent claim on a single asset written as

$$\frac{\partial C}{\partial t} + e\,(s,t)\frac{\partial^2 C}{\partial S^2} + p\,(s,t)\frac{\partial C}{\partial S} + h\,(s,t)C = 0 \tag{10}$$

where t either represents calendar time or time-to-expiry, s represents either the value of the underlying asset or some monotonic function of it (e.g. $log\ (s_t)$; $\log - spot$) and C is the value of the claim (as a function of s and t). The term $e(.), p(.)$ and $h(.)$ are the diffusion, convection and reaction coefficient respectively, and this type of PDE is known as a convection-diffusion PDE. This type of PDE can also be written in the form

$$\frac{\partial C}{\partial t} + e\,(s.t)\frac{\partial}{\partial S}\left(\alpha(s,t)\frac{\partial C}{\partial S}\right) + p(s,t)\frac{\partial}{\partial S}(\beta(s,t)C) + h(s,t)c = 0. \tag{11}$$

This form occurs in the Fokker-Planck (Kolmogrov forward) equation that describes the evolution of the transition density of a stochastic quantity (e.g. a stock value). This can be put in the form of equation (10) if the functions α and β are both differentiable in S-although it is usually better to directly discretize the form given [11].

We do not consider this form (11) further in the paper. Equation (10) can be solved with Dirichlet boundary conditions and initial condition (see [12,10]).

Dirichlet boundary conditions, which take the form.

$$C\,(s,t) = g\,(s,t) \forall\, s \,\epsilon\, \Gamma(t) \tag{12}$$

where $\Gamma(t)$ is the (possibly time-dependent) boundary of the region. In the ID case this simplifies to

$$C(s_P(t), t) = g(t) \tag{13}$$

Initial condition

$$C(s, 0) = g(s) \tag{14}$$

to some terminal time T, so the time domain is naturally bound.

Note that failure to provide consistent boundary condition will greatly reduce accuracy of the solution.

To begin with, we just consider discretization in the spatial dimension. Given the time-dependent vector $C(t) = \left(C(s_0, t), C(s_{1,}t) \dots\dots.. C(S_N, t), C(S_{N+1,}t)\right)^T$, the derivatives are given by

$$\frac{\partial C}{\partial S} \approx D^1 C(t) \tag{15}$$

$$\frac{d^2C}{dS^2} \approx D^2 C(t)$$

If we similarly define $e(t) = \left(e(s_0, t), e(s_1, t), \dots, e(S_N, t), \ e\,(S_{N+1}, t)\right)^T$, and likewise for $p(t)$ and $h(t)$, then equation (10) may be written as a set of N+2 coupled ordinary differential equations (ODEs)

$$\frac{dC(t)}{dt} = -L(t)\, C(t) \tag{16}$$

where $L(t) \equiv E(t)D^2 + P(t)D^1 + H(t) and\, E(t) = diag\,(e(t))$ etc (i.e. the diagonal matrix with the elements of e along the lead diagonal).

In general equation (16) can be denoted in the form

$$\frac{dx}{dt} = -f\,(x_t\ ; \alpha) \tag{17}$$

where $x(t) \in R^n, and\ \alpha \in R$ is a parameter. This usual notation x_t denotes the values of the system state over a time windows of finite length τ, that is $x_t(\theta) = x\,(t + \theta) \in R^n, \theta \in [-\tau, 0]$, and $x_t \in C$, where $C \coloneqq c\,([-\tau, o], R^n)$ denotes the Banach space of continuous functions over the interval $[-\tau, o]$ equipped with the supremum norm. It is assumed that $f: C\ x\ R \longrightarrow R^n$ is twice continuously differentiable in its arguments and $f(o; \alpha) = 0$ for all α. Assume further that the origin undergoes a supercritical Hopf bifurcation at $\alpha = 0$. Hence, for small positive α the origin is unstable and there exist a small amplitude limit cycle. To study the behavior near the origin, it is convenient to scale the variable $x \longrightarrow \varepsilon x\ and\ \alpha \longrightarrow \varepsilon \alpha$, where ε is a small positive parameter. This transforms (17) into a weekly nonlinear system of the form

$$\frac{dx}{dt} = -\left(L\, x_t + \varepsilon\, f\,(x_t;\, \varepsilon)\right) \tag{18}$$

where $L: C \longrightarrow R^n$is a linear operator and f is a C^2 function with$f(o; \varepsilon) = o$ for all ε. Equation (18) is a perturbation of the linear equation [1].

$$dx/_{dt} = -L\, x_t \tag{19}$$

5 Multiple Scale and Financial Derivatives

In this section, we show how the method of multiple scale can be applied to the discretized PDE in equation (19).

Example (2)

Consider the DDE problem, one that has an exact solution, namely;

$$\frac{dx}{dt} = -x(t-T), T = {}^{\pi}/_{2} + \varepsilon\mu \quad . \tag{20}$$

Equation (20) undergoes a Hopf bifurcation at $\varepsilon = o$, because the following conditions in example 1 hold. Hence, equation (20) behaves similar to Black-Scholes differential equation involving a term $X''(s)$.

Lemma 1

Let t be replaced by two time variable: regular time B (A riskless Bond (cash) as in (5)) and slow time $S = \varepsilon t$ (underlying security which evolves in accordance with stock price S, as in (6)), then the solution of (20) is given by

$$X_o = R_o \exp\left(\frac{4\mu(S_o e^{\left(\mu - \sigma^2/_2\right)t + \sigma w_t}}{\pi^2 + 4}\right) \cos\left(r^{-1} \ln BB_0{}^{-1} - \left(\frac{2\pi\mu\left(S_o e^{\left(\mu - \sigma^2/_2\right)t + \sigma w_t}\right)}{\pi^2 + 4} + \theta_0\right)\right) \tag{21}$$

Proof

Let the dependent variable $x(t)$ be replace by $x\ (B, S)$. Hence, if $x\ (B, S)$

$$dx = \frac{\partial x}{\partial B}\ dB + \frac{\partial x}{\partial s}\ ds$$
$$\frac{dx}{dt} = \frac{\partial x}{\partial B}\frac{dB}{dt} + \frac{\partial x}{\partial s}\frac{ds}{dt}$$
$$\frac{dx}{dt} = \frac{\partial x}{\partial B} + \varepsilon\frac{\partial x}{\partial s}.$$

From equation (20), we have

$$\frac{\partial x}{\partial B} + \varepsilon\frac{\partial x}{\partial s} = -x(B - T, S - \varepsilon T) \tag{22}$$

Since $T = {}^{\pi}/_{2} + \varepsilon\mu$, the delayed term may be expanded for small ε as follows;

$$x\left(B - {}^{\pi}/_{2} - \epsilon\mu, S - \varepsilon\, {}^{\pi}/_{2} - \varepsilon^2\mu\right)$$
$$= x\left(\beta - {}^{\pi}/_{2}, S\right) - \varepsilon\,\mu\,\frac{\partial x_d}{\partial B} - \varepsilon\,{}^{\pi}/_{2}\frac{\partial x_d}{\partial S} + 0\ (\varepsilon^2) \tag{23}$$

where x_d is an abbreviation for $x\ \left(B - {}^{\pi}/_{2}, S\right)$. Next we expand

$$x = x_o + \varepsilon x_1 + 0\ (\varepsilon^2), \tag{24}$$

So

$$\frac{\partial x}{\partial B} = \frac{\partial x_o}{\partial B} + \varepsilon \frac{\partial x_1}{\partial B} + 0(\varepsilon^2) \quad (25)$$

$$\frac{\partial x}{\partial S} = \frac{\partial x_o}{\partial S} + \varepsilon \frac{\partial x_1}{\partial S} + 0(\varepsilon^2) \quad . \quad (26)$$

By substituting equation (23), (25) and (26) into (22), gives

$$\frac{\partial x_o}{\partial B} + \varepsilon \frac{\partial x_1}{\partial B} + \varepsilon \frac{\partial x_o}{\partial S} + \varepsilon^2 \frac{\partial x_1}{\partial S} = -x_o\left(B - {}^{\pi}/_2, S\right)$$
$$- \varepsilon x_1\left(B - {}^{\pi}/_2, S\right) + \varepsilon\mu \frac{\partial x_{od}}{\partial B} + \varepsilon^2 \mu \frac{\partial x_{1d}}{\partial B} \ \varepsilon {}^{\pi}/_2 \frac{\partial x_{0d}}{\partial S}$$
$$+ \varepsilon^2 {}^{\pi}/_2 \frac{\partial x_{1d}}{\partial S} + 0(\varepsilon^2) \quad (27)$$

$$\frac{\partial x_o}{\partial B} + x_o\left(B - {}^{\pi}/_2, S\right) = 0 \quad (28)$$

$$\frac{\partial x_1}{\partial B} + x_1\left(B - {}^{\pi}/_2, S\right) = \mu \frac{\partial x_{0d}}{\partial B} + {}^{\pi}/_2 \frac{\partial x_{od}}{\partial S} - \frac{\partial x_o}{\partial S} \quad (29)$$

Equation (28) has periodic solution (since (20) is autonomous)

$$x_o = R\,(S)\cos(B - \theta(S)) \quad (30)$$

where as usual in this method $R(S)$ (the approximated amplitude of periodic motion of stock) and $\theta(S)$ (the frequency of the bifurcated periodic solution) are yet undetermined function of slow times S.

Taken $x_{0d} = -\frac{\partial x_0}{\partial B}$ in equation (29)

$$\frac{\partial x_1}{\partial B} + x_1\left(B - {}^{\pi}/_2, S\right) = -\frac{\mu \partial^2 x_o}{\partial B^2} - {}^{\pi}/_2 \frac{\partial^2 x_o}{\partial B \partial S} - \frac{\partial x_o}{\partial S} \ . \quad (31)$$

Substitute equation (30) into (31)

$$\frac{\partial x_o}{\partial B} = -R\sin(B - \theta)$$
$$\frac{\partial x_o}{\partial S} = R\theta' \sin(B - \theta) + R' \cos\ (B - \theta)$$
$$\frac{\partial^2 x_o}{\partial B^2} = -R\ cos\ (B - \theta)$$
$$\frac{\partial^2 x_o}{\partial B \partial S} = -[-R\theta' \cos(B - \theta) + R' \sin(B - \theta)]$$
$$= R\theta' \cos(B - \theta) - R' \sin(B - \theta)\ .$$

Let $B - \theta = \psi, by\ substituting\ \frac{\partial x_o}{\partial B}, \frac{\partial x_o}{\partial S}, \frac{\partial^2 x_o}{\partial B^2}\ and\ \frac{\partial^2 x_o}{\partial B \partial S}$ in (31), we have

$$\mu R\cos(\psi) - {}^{\pi}/_2 R\theta' \cos(\psi) + {}^{\pi}/_2 R' \sin(\psi) - R\theta' \sin(\psi) - R' \cos(\psi) = \frac{\partial x_1}{\partial B} + x_1\left(B - {}^{\pi}/_2, S\right) \quad (32)$$

Equating coefficient of sin (ψ) and cos (ψ) to zero

$$\mu R - {}^{\pi}/_2 R\theta' - R' = 0 \quad (33)$$

$$\pi/_2 R' - R\theta' = 0 \quad . \tag{34}$$

From equation (34)

$$\theta' = \frac{\pi R'}{2R} \quad . \tag{35}$$

Plug equation (35) in equation (34)

$$R' = \frac{4\mu R}{\pi^2 + 4} \quad . \tag{36}$$

Plug (36) in equation (35)

$$\theta' = \frac{2\pi\mu}{\pi^2 + 4} \quad . \tag{37}$$

From equation (36), we get

$$\frac{1}{R}\, dR = \frac{4\mu}{\pi^2 + 4}\, ds$$

so that

$$\ln R = \frac{4\mu S}{\pi^2 + 4} + K,$$

and

$$R(S) = R_o \exp\left(\frac{4\mu S}{\pi^2 + 4}\right) \quad . \tag{38 }$$

Similarly

$$\theta\,(S) = \frac{2\pi\mu S}{\pi^2 + 4} + \theta_o \tag{39}$$

substitute R(s) and θ (s) in equation (30)

$$x \approx x_o = R_o \exp\left(\frac{4\mu S}{\pi^2 + 4}\right) \cos\left(t - \left(\frac{2\pi\mu S}{\pi^2 + 4} + \theta_o\right)\right). \tag{40}$$

It is not difficult to show from (5) and (6) that

$$B = B_0 e^{rt}, \tag{41}$$

and

$$S = S_o e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\, w_t} \tag{42}$$

respectively.

Solving for t in (41) and plugging it in (40) using (42), we have (21) as required.

6 Conclusion

The periodic solution derived from two way variable expansion method (21) proves that, if the time is delayed, the oscillatory movement of the price of stock can be monitored and instability controlled and stabilized. Using the slow time $S = \varepsilon t$ and equation (42), we have

$$S_0 = \varepsilon t exp\left[\left(\frac{\sigma^2}{2} - \mu\right)t - \sigma W_t\right]. \tag{43}$$

Plugging (43) into (21) we arrive at;

$$X_o = R_o \exp\left(\frac{4\mu\varepsilon t}{\pi^2+4}\right)\cos\left(r^{-1}\ln BB_0{}^{-1} - \left(\frac{2\pi\mu\varepsilon t}{\pi^2+4} + \theta_0\right)\right). \tag{44}$$

Notice that $S_0 \to 0$ as $\varepsilon \to 0$ and X_0 fluctuate according to r.

Competing Interests

Authors have declared that no competing interests exist.

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