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Deformation of Surfaces in Three-Dimensional Space Induced by Means of Integrable Systems

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Research Article

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Abstract

The correspondence between different versions of the Gauss-Weingarten equation is investigated. The compatibility condition for one version of the Gauss-Weingarten equation gives the Gauss-Mainardi-Codazzi system. A deformation of the surface is postulated which takes the same form as the original system but contains an evolution parameter. The compatibility condition of this new augmented system gives the deformed Gauss-Mainardi-Codazzi system. A Lax representation in terms of a spectral parameter associated with the deformed system is established. Several important examples of integrable equations based on the deformed system are then obtained. It is shown that the Gauss-Mainardi-Codazzi system can be obtained as a type of reduction of the self-dual Yang-Mills equations.

Keywords: Integrable system, surface, Lax representation, deformation, Gauss-Weingarten equation
MSCs: 35A30, 32A25, 35C05

1 Introduction

There are a great many phenomena in nature which make use of the concept of a surface in formulating a realistic and useful model which accounts for the observations and properties which are of interest to investigate (Rogers and Schief, 2002). Quite frequently, it is required by the nature of the circumstances that these surfaces evolve over the course of time, as opposed to remaining completely static. An example of the former would be a propagating shock front, and of the latter, a surface formed by the surface tension at the interface between two different liquids. In the course of the development of this kind of theory, there occurs as a consequence of the method or process used the appearance of various kinds of nonlinear partial differential equations in a natural way. As a result, the study of all aspects of these equations becomes an important topic in itself, and becomes linked to the study of the evolution problem. In the larger picture, there results an interaction between the areas which concern the differential geometry of surfaces (Spivak, 1999) and nonlinear partial differential

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equations which arise in the course of this work (Ablowitz and Clarkson, 1992; Das, 1989; Tenenblat, 1998). This kind of interaction has been of mutual benefit to the development of these subjects.

Consequently, many nonlinear phenomena in physics, which can be described by various kinds of partial differential equation, are also closely related to the evolution of surfaces with respect to an evolution parameter such as time (Bracken, 2010). Moreover, it has been found that these types of nonlinear equations possess solitary wave solutions (Sasaki, 1979; Martina et al., 2001). Thus, there are a great number of links between many diverse areas and this is largely based on the fact that a great many of the local properties of surfaces can be expressed in the form of nonlinear partial differential equations. For example, two equations which have played a particularly important role in the development of the subject are the sine-Gordon and Liouville equations, respectively. These equations, especially the former, have also played a prominent role in the development of Bäcklund transformations. In fact, a generic method for the description of soliton interaction has its roots in a type of transformation originally introduced by Bäcklund to generate pseudospherical surfaces (Chern and Tenenblat, 1986; Bracken, 2009). The sine-Gordon equation was generated in the nineteenth century from the Gauss-Mainardi-Codazzi system for pseudo-spherical surfaces. This equation was subsequently rederived independently by both Enneper and Bonnet in a similar way. A purely geometric construction for pseudospherical surfaces was reformulated as a transformation by Bianchi later. The interrelationship between deformations of surfaces and integrable systems in $2+1$ dimensions has been discussed by many researchers (Konopelchenko, 1993; Ablowitz and Chakravarty, 1993).

The objective here is to investigate the deformation of surfaces and the relationship of this topic to the study of various aspects of integrable systems in various dimensions. It will be seen that many integrable $(2+1)$-dimensional nonlinear partial differential equations can be obtained from the $(2+1)$-dimensional Gauss-Mainardi-Codazzi equation, which can be interpreted as describing the deformation, or motion, of a surface. It is hoped that the discussion will benefit from the new accompanying proofs.

A particularly remarkable example which is to be studied is that of the reduction of the self-dual Yang-Mills equation to Gauss-Mainardi-Codazzi form. Yang-Mills systems have numerous applications in particle physics. It may be said that the self-dual Yang-Mills system appears to be a universal integrable system from which many other integrable equations can be obtained by symmetry reductions and specification of the Lie algebra. In fact, it has been conjectured by Ward (Ward, 1973) that all integrable $(1+1)$-dimensional nonlinear differential equations may be obtained by reduction directly from the self-dual Yang-Mills equations. In fact, many soliton equations in $(2+1)$ dimensions have been found as reductions of the same self-dual system, and some new ones appear here. Finally, it will be shown that the linear systems introduced here give rise to a number of integrable equations which are well known and of interest. These equations are developed as a result of specific reductions of the Gauss-Mainardi-Codazzi system.

2 Surface Theory and the Gauss Weingarten Equation

Let M be a smooth manifold or surface in $\mathbb{R}^3$ with a local coordinate system specified by (x, y). Let $\mathbf{r} = \mathbf{r}(x, y)$ denote the position vector of a generic point P on M in $\mathbb{R}^3$. The vectors $\mathbf{r}_x$ and $\mathbf{r}_y$ are tangential to M at P, and at points where they are linearly independent

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}, \tag{2.1}$$

determine a unit normal to M. The first and second fundamental forms of this surface are given by

$$I = d\mathbf{r}^2 = E\,dx^2 + 2F\,dxdy + G\,dy^2, \tag{2.2}$$

$$II = d\mathbf{r} \cdot \mathbf{n} = L\,dx^2 + 2M\,dxdy + N\,dy^2, \tag{2.3}$$

where the coefficient terms are defined to be,

$$E = \mathbf{r}_x^2, \qquad F = \mathbf{r}_x \cdot \mathbf{r}_y, \qquad G = \mathbf{r}_y^2, \tag{2.4}$$

$$L = \mathbf{r}_{xx} \cdot \mathbf{n}, \qquad M = \mathbf{r}_{yx} \cdot \mathbf{n}, \qquad N = \mathbf{r}_{yy} \cdot \mathbf{n}. \tag{2.5}$$

An important classical result due to Bonnet states that the sextuplet $\{E, F, G, L, M, N\}$ determines M up to its position in space. There is a third fundamental form which does not depend on the choice of $\mathbf{n}$ and does not contain much beyond what is prescribed by (2.2)-(2.3) since it is expressible in terms of I and II as

$$III = d\mathbf{n} \cdot d\mathbf{n} = 2H \cdot II - K \cdot I. \tag{2.6}$$

In (2.6), K, H are the gaussian and mean curvatures of M, respectively.

The Gauss equations associated with M are

$$\mathbf{r}_{xx} = \Gamma^1_{11}\mathbf{r}_x + \Gamma^2_{11}\mathbf{r}_y + L\mathbf{n}, \qquad \mathbf{r}_{xy} = \Gamma^1_{12}\mathbf{r}_x + \Gamma^2_{12}\mathbf{r}_y + M\mathbf{n}, \qquad \mathbf{r}_{yy} = \Gamma^1_{22}\mathbf{r}_x + \Gamma^2_{22}\mathbf{r}_y + N\mathbf{n}, \tag{2.7}$$

while the Weingarten equations are given as

$$\mathbf{n}_x = P^1_1\,\mathbf{r}_x + P^2_1\mathbf{r}_y, \qquad \mathbf{n}_y = P^1_2\mathbf{r}_x + P^2_2\mathbf{r}_y. \tag{2.8}$$

The ten coefficient functions in systems (2.7)-(2.8) are given in terms of the sixtuplet $\{E, F, G, L, M, N\}$ as follows,

$$\begin{gathered}
\Gamma^1_{11} = \frac{1}{2g}(GE_x - 2FF_x + FE_y), \qquad \Gamma^2_{11} = \frac{1}{2g}(2EF_x - EE_y - FE_x), \\
\Gamma^1_{12} = \frac{1}{2g}(GE_y - FG_x), \qquad \Gamma^2_{12} = \frac{1}{2g}(EG_x - FE_y), \\
\Gamma^1_{22} = \frac{1}{2g}(2GF_y - GG_x - FG_y), \qquad \Gamma^2_{22} = \frac{1}{2g}(EG_y - 2FF_y + FG_x),
\end{gathered} \tag{2.9}$$

$$P^1_1 = \frac{MF - LG}{g}, \quad P^2_1 = \frac{LF - ME}{g}, \quad P^1_2 = \frac{NF - MG}{g}, \quad P^2_2 = \frac{MF - NE}{g}, \tag{2.10}$$

where

$$g = |\mathbf{r}_x \times \mathbf{r}_y|^2 = EG - F^2. \tag{2.11}$$

The Γ^i_{jk} in (2.9) are the usual Christoffel symbols given by the expression

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}), \tag{2.12}$$

where, upon setting $x^1 = x$ and $x^2 = y$, the first fundamental form (2.2) becomes

$$I = g_{jk}\,dx^j\,dx^k,$$

where

$$g^{jk} g_{kl} = \delta^j_l.$$

The compatibility condition applied to the linear system (2.2) produces the nonlinear Mainardi-Codazzi system

$$L_y - M_x = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}, \qquad M_y - N_x = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}. \tag{2.13}$$

An equivalent and perhaps more elegant way of representing the Gauss and Weingarten equations is to write them as a linear system. To this end, an orthogonal basis on M can be easily established by defining

$$\mathbf{e}_1 = \frac{\mathbf{r}_x}{\sqrt{E}}, \qquad \mathbf{e}_2 = \mathbf{n}, \qquad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2, \tag{2.14}$$

and moreover,

$$\mathbf{r}_y = \frac{F}{\sqrt{E}}\mathbf{e}_1 - \sqrt{\frac{g}{E}}\mathbf{e}_3. \tag{2.15}$$

Using (2.14) and (2.15), $\mathbf{e}_3$ can be expressed in terms of $\mathbf{r}_x$ and $\mathbf{r}_y$ as

$$\mathbf{e}_3 = \sqrt{\frac{E}{g}}(\frac{F}{E}\mathbf{r}_x - \mathbf{r}_y). \tag{2.16}$$

Theorem 2.1. The Gauss-Weingarten system of equations (2.9)-(2.10) are exactly equivalent to the linear system defined by

$$\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}_x = \mathbf{A}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \qquad \begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}_y = \mathbf{C}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}. \tag{2.17}$$

The matrices $\mathbf{A}$ and $\mathbf{C}$ are defined to be

$$\mathbf{A} = \begin{pmatrix} 0 & \kappa & -\sigma\\ -\kappa & 0 & \tau\\ \sigma & -\tau & 0\end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & \omega_3 & -\omega_2\\ -\omega_3 & 0 & \omega_1\\ \omega_2 & -\omega_1 & 0\end{pmatrix} \tag{2.18}$$

The entries of the matrices $\mathbf{A}$ and $\mathbf{C}$ are given by the following expressions

$$\kappa = \frac{L}{\sqrt{E}}, \qquad \tau = -\sqrt{\frac{g}{E}}P_1^2, \qquad \sigma = \frac{\sqrt{g}}{E}\Gamma_{11}^2, \tag{2.19}$$

and

$$\omega_1 = -\sqrt{\frac{g}{E}}P_2^2, \qquad \omega_2 = \frac{\sqrt{g}}{E}\Gamma_{12}^2, \qquad \omega_3 = \frac{M}{\sqrt{E}}. \tag{2.20}$$

Proof: The top element of the first matrix equation in (2.17) using (2.14) is given by

$$(\frac{\mathbf{r}_x}{\sqrt{E}})_x = \kappa\mathbf{n} - \sigma\mathbf{e}_3.$$

Expanding the derivative on the left-hand side and solving for $\mathbf{r}_{xx}$, there results the expression,

$$\mathbf{r}_{xx} = \frac{1}{E}(\frac{E_x}{2} - \Gamma_{11}^2 F)\mathbf{r}_x + \Gamma_{11}^2\mathbf{r}_y + L\mathbf{n}. \tag{2.21}$$

Substituting Γ_{11}^2 from (2.9) into (2.21) and then simplifying the coefficient of $\mathbf{r}_x$, we obtain,

$$\mathbf{r}_{xx} = \frac{1}{2g}(GE_x + FE_y - 2FF_x)\mathbf{r}_x + \Gamma_{11}^2\mathbf{r}_y + L\mathbf{n} = \Gamma_{11}^1\mathbf{r}_x + \Gamma_{11}^2\mathbf{r}_y + L\mathbf{n}.$$

This is the first equation in the set (2.7).

The second element in the first equation of (2.17) is given by

$$\mathbf{n}_x = -\frac{L}{E}\mathbf{r}_x + \tau\mathbf{e}_3 = -\frac{L}{E}\mathbf{r}_x - (\frac{LF-ME}{g})\frac{F}{E}\mathbf{r}_x + \frac{LF-ME}{g}\mathbf{r}_y$$

$$= -\frac{1}{gE}(gL - LF^2 + MEF)\mathbf{r}_x + P_1^2\mathbf{r}_y = P_1^1\mathbf{r}_x + P_1^2\mathbf{r}_y.$$

This is the first equation in (2.8).

From the second system in (2.17), the first entry is

$$(\frac{\mathbf{r}_x}{\sqrt{E}})_y = \omega_3\mathbf{n} - \omega_2\mathbf{e}_3.$$

Developing the derivative and then solving for the derivative $\mathbf{r}_{xy}$, we obtain

$$\mathbf{r}_{xy} = (\frac{E_y}{2E} - \frac{F}{E}\Gamma_{12}^2)\mathbf{r}_x + \Gamma_{12}^2\mathbf{r}_y + M\mathbf{n}.$$

Substituting Γ_{12}^2 from (2.9) and simplifying gives

$$\mathbf{r}_{xy} = \frac{1}{2gE}(E_yEG - E_yF^2 - FEG_x + F^2E_y)\mathbf{r}_x + \Gamma_{12}^2\mathbf{r}_y + M\mathbf{n} = \frac{1}{2g}(E_yG - FG_x)\mathbf{r}_x + \Gamma_{12}^2\mathbf{r}_y + M\mathbf{n}$$

$$= \Gamma_{12}^1\mathbf{r}_x + \Gamma_{12}^2\mathbf{r}_y + M\mathbf{n}.$$

This is the second equation of (2.7).

The second element of the $\mathbf{C}$ system in (2.17) yields,

$$\mathbf{n}_y = -\omega_3\frac{\mathbf{r}_x}{\sqrt{E}} + \omega_1\mathbf{e}_3 = -\frac{M}{E}\mathbf{r}_x - \sqrt{\frac{g}{E}}P_2^2\sqrt{\frac{E}{g}}(\frac{F}{E}\mathbf{r}_x - \mathbf{r}_y) = -\frac{1}{E}(M + FP_2^2)\mathbf{r}_x + P_2^2\mathbf{r}_y$$

$$= -\frac{1}{gE}(MEG - MF^2 + MF^2 - NEF)\mathbf{r}_x + P_2^2\mathbf{r}_y = P_2^1\mathbf{r}_x + P_2^2\mathbf{r}_y.$$

This is the second of the equations in (2.8). Four of the equations have been obtained, it remains to get $\mathbf{r}_{yy}$.

The third element of the second system is $\mathbf{e}_{3y} = \omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2$. Differentiating this and solving for $\mathbf{r}_{yy}$,

$$\mathbf{r}_{yy} = (\frac{F}{E})_y\mathbf{r}_x + \frac{F}{E}\mathbf{r}_{xy} - (\sqrt{\frac{g}{E}})_y\mathbf{e}_3 - \frac{g}{E^2}\Gamma_{12}^2\mathbf{r}_x - \frac{g}{E}P_2^2\mathbf{e}_2.$$

Since $\mathbf{r}_{xy}$ has already been determined in this process, we substitute $\mathbf{r}_{xy}$ above and $\mathbf{e}_3$ from (2.16) to complete the set,

$$\mathbf{r}_{yy} = [(\frac{F}{E})_y + \frac{F}{E}\Gamma_{12}^1 - (\frac{g}{E})_y(\frac{E}{g})^{1/2}\frac{F}{E} - \frac{g}{E^2}\Gamma_{12}^2]\mathbf{r}_x + [\frac{F}{E}\Gamma_{12}^2 + (\frac{g}{E})_y(\frac{E}{g})^{1/2}]\mathbf{r}_y + [M\frac{F}{E} - \frac{g}{E}P_2^2]\mathbf{n}.$$

$$= \frac{1}{2g}(2EF_y - FG_y - GG_x)\mathbf{r}_x + \frac{1}{2g}(EG_y + FG_x - 2FF_y)\mathbf{r}_y + N\mathbf{n} = \Gamma_{22}^1\mathbf{r}_x + \Gamma_{22}^2\mathbf{r}_y + N\mathbf{n}.$$

This completes the proof.

Theorem 2.2. The compatibility condition for system (2.17) gives a relationship between $\mathbf{A}$ and $\mathbf{C}$ of the form,

$$\mathbf{A}_y - \mathbf{C}_x + [\mathbf{A}, \mathbf{C}] = 0. \tag{2.22}$$

Equation (2.22) in terms of the matrix elements (2.18) is equivalent to the system

$$\kappa_y - \omega_{3x} + \sigma\omega_1 - \tau\omega_2 = 0, \quad \sigma_y - \omega_{2x} - \kappa\omega_1 + \tau\omega_3 = 0, \quad \tau_y - \omega_{1x} + \kappa\omega_2 - \sigma\omega_3 = 0. \tag{2.23}$$

Proof: Differentiate the first equation of (2.17) with respect to y and the second with respect to x. Requiring that the mixed partial derivatives be equal yields (2.22). By calculating (2.22) using the matrices in (2.18) immediately yields (2.23).

The matrix system in Theorem 2.1 can be mapped onto a 2×2 system which gives rise to exactly the same set (2.23).

Theorem 2.3. Linear system (2.17)-(2.18) which satisfies (2.22) or explicitly (2.23) can also be expressed in the form of a 2×2 matrix system given by

$$\psi_x = \mathbf{U}\psi, \qquad \psi_y = \mathbf{V}\psi. \tag{2.24}$$

If σ_i are the usual Pauli matrices, then $\mathbf{U}$ and $\mathbf{V}$ in (2.24) can be expressed as

$$\mathbf{U} = \frac{1}{2i}(\kappa\sigma_1 - \sigma\sigma_2 + \tau\sigma_3), \qquad \mathbf{V} = \frac{1}{2i}(\omega_3\sigma_1 - \omega_2\sigma_2 + \omega_1\sigma_3).$$

Explicitly, $\mathbf{U}$ and $\mathbf{V}$ take the form,

$$\mathbf{U} = \frac{1}{2i}\begin{pmatrix} \tau & \kappa + i\sigma \\ \kappa - i\sigma & -\tau \end{pmatrix}, \qquad \mathbf{V} = \frac{1}{2i}\begin{pmatrix} \omega_1 & \omega_3 + i\omega_2 \\ \omega_3 - i\omega_2 & -\omega_1 \end{pmatrix}. \tag{2.25}$$

Proof: Following the same procedure used in Theorem 2.2, the compatibility condition for (2.25) is given by

$$\mathbf{U}_y - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0. \tag{2.26}$$

Explicitly, the commutator takes the form

$$\mathbf{UV} - \mathbf{VU} = -\frac{1}{2}\begin{pmatrix} -i(\kappa\omega_2 - \sigma\omega_3) & \tau(\omega_3 + i\omega_2) - \omega_1(\kappa + i\sigma) \\ \omega_1(\kappa - i\sigma) - \tau(\omega_3 - i\omega_2) & i(\kappa\omega_2 - \sigma\omega_3) \end{pmatrix}.$$

The diagonal element of (2.26) gives the third equation in (2.23) and the two off-diagonal elements are

$$\kappa_y + i\sigma_y - \omega_{3x} - i\omega_{2x} + i\tau(\omega_3 + i\omega_2) - i\omega_1(\kappa + i\sigma) = 0,$$

$$\kappa_y - i\sigma_y - \omega_{3x} + i\omega_{2x} + i\omega_1(\kappa - i\sigma) - i\tau(\omega_3 - i\omega_2) = 0.$$

Upon adding and subtracting these two equations, the remaining pair in (2.23) result.

3 Deformations of Surfaces

At this point, an additional parameter is introduced into the picture. This parameter can be interpreted as an evolution or time parameter. Thus the deformation of a surface will be described with respect to a new, additional parameter t. To carry this out, it is postulated that such a deformation or motion of a surface is governed by the following system

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = \mathbf{A}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \mathbf{B}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = \mathbf{C}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{3.1}$$

In (3.1), the matrices $\mathbf{A}$ and $\mathbf{C}$ are given by (2.18), whereas matrix $\mathbf{B}$ is a new matrix which has the form,

$$\mathbf{B} = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}. \tag{3.2}$$

The γ_i in (3.2) are real functions of the given variables. System (3.1) will be referred to as the deformed or $(2+1)$-dimensional Gauss-Weingarten system. Proceeding in exactly the same way as was done in Theorem 2.2, that theorem can be extended to include deformations.

Theorem 3.1. The compatibility conditions applied to the deformed Gauss-Weingarten system (3.1) generates the $(2+1)$-dimensional augmented Gauss-Mainardi-Codazzi equations in the form

$$\mathbf{A}_y - \mathbf{C}_x + [\mathbf{A}, \mathbf{C}] = 0, \qquad \mathbf{A}_t - \mathbf{B}_x + [\mathbf{A}, \mathbf{B}] = 0, \qquad \mathbf{B}_y - \mathbf{C}_t + [\mathbf{B}, \mathbf{C}] = 0. \tag{3.3}$$

Based on (2.18) and (3.2), system (3.3) is equivalent to the following set of nine equations,

$$\kappa_y - \omega_{3x} + \sigma\omega_1 - \tau\omega_2 = 0, \qquad \sigma_y - \omega_{2x} + \tau\omega_3 - \kappa\omega_1 = 0, \qquad \tau_y - \omega_{1x} + \kappa\omega_2 - \sigma\omega_3 = 0,$$
$$\kappa_t - \gamma_{3x} + \sigma\gamma_1 - \tau\gamma_2 = 0, \qquad \sigma_t - \gamma_{2x} + \tau\gamma_3 - \kappa\gamma_1 = 0, \qquad \tau_t - \gamma_{1x} + \kappa\gamma_2 - \sigma\gamma_3 = 0,$$
$$\gamma_{3y} - \omega_{3t} + \gamma_2\omega_1 - \gamma_1\omega_2 = 0, \quad \gamma_{2y} - \omega_{2t} + \gamma_1\omega_3 - \gamma_3\omega_1 = 0, \quad \gamma_{1y} - \omega_{1t} + \gamma_3\omega_2 - \gamma_2\omega_3 = 0. \tag{3.4}$$

It is just a direct calculation to develop (3.4) and will not be included. The first equation of (3.3) just reproduces the result in (2.22)-(2.23).

It is remarkable to realize that there is a Lax representation or linear problem associated with system (3.3). To be able to write it down, it is first useful to introduce some notation in which to formulate the equations. To compress the result somewhat, introduce the complex variable $z = \frac{1}{2}(x + it)$ and its conjugate $\bar{z}$ as well as the new matrices $\mathbf{F}^{\pm}$ defined to be

$$\mathbf{F}^{\pm} = \mathbf{A} \pm i\mathbf{B}. \tag{3.5}$$

With this in mind, the following important result can be given.

Theorem 3.2. There exists a Lax representation for system (3.3) of the form,

$$\Psi_z = \tfrac{|\lambda|^2}{1-|\lambda|^4}(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi + \tfrac{1}{1-|\lambda|^4}(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi,$$
$$\Psi_{\bar{z}} = \tfrac{|\lambda|^2}{1-|\lambda|^4}(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi + \tfrac{1}{1-|\lambda|^4}(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi, \tag{3.6}$$
$$\Psi_y = -i\tfrac{\lambda|\lambda|^2}{1-|\lambda|^4}(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi - i\tfrac{\lambda}{1-|\lambda|^4}(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi + (\mathbf{C} + i\lambda\mathbf{F}^+)\Psi.$$

In (3.6), λ is a complex spectral parameter. The compatibility conditions for system (3.6) reproduce system (3.3) exactly.

Proof: The proof is just a matter of a long calculation. In fact symbolic manipulation has been used to do it. It suffices to say that noncommutative multiplication tables, which are defined in terms of the noncommuting elements that appear here, can be formulated within *Maple* to deal with noncommuting terms that arise. In this way, the compatibilty condition can be worked out efficiently. One of the three compatibility conditions will be developed here, the other two are carried out in an identical manner. Only some of the steps will be given explicitly.

Beginning with Ψ_z in (3.6), it is differentiated with respect to $\bar{z}$ to obtain,

$$(1-|\lambda|^4)\Psi_{z\bar{z}} = |\lambda|^2(\mathbf{F}^+_{\bar{z}} - |\lambda|^2\mathbf{F}^-_{\bar{z}})\Psi + |\lambda|^2(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi_{\bar{z}} + (\mathbf{F}^-_{\bar{z}} - |\lambda|^2\mathbf{F}^+_{\bar{z}})\Psi + (\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi_{\bar{z}}.$$

At this point, the derivative $\Psi_{\bar{z}}$ given in (3.6) can be substituted into this expression to produce,

$$(1-|\lambda|^4)\Psi_{z\bar{z}} = |\lambda|^2(\mathbf{F}^+_{\bar{z}} - |\lambda|^2\mathbf{F}^-_{\bar{z}})\Psi + (\mathbf{F}^-_{\bar{z}} - |\lambda|^2\mathbf{F}^+_{\bar{z}})\Psi + \frac{|\lambda|^4}{1-|\lambda|^4}(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi$$
$$+\frac{|\lambda|^2}{1-|\lambda|^4}(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi + \frac{|\lambda|^2}{1-|\lambda|^4}(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi \tag{3.7}$$
$$+\frac{1}{1-|\lambda|^4}(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi.$$

Similarly, by differentiating $\Psi_{\bar{z}}$ in (3.6) with respect to z and replacing Ψ_z, we obtain

$$(1-|\lambda|^4)\Psi_{\bar{z}z} = |\lambda|^2(\mathbf{F}^-_z - |\lambda|^2\mathbf{F}^+_z)\Psi + (\mathbf{F}^+_z - |\lambda|^2\mathbf{F}^-_z)\Psi + \frac{|\lambda|^4}{1-|\lambda|^4}(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi$$

$$+\frac{|\lambda|^2}{1-|\lambda|^4}(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi + \frac{|\lambda|^2}{1-|\lambda|^4}(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)\Psi$$
$$+\frac{1}{1-|\lambda|^4}(\mathbf{F}^+ - |\lambda|^2\mathbf{F}^-)(\mathbf{F}^- - |\lambda|^2\mathbf{F}^+)\Psi \qquad (3.8)$$

Using (3.7) and (3.8) in the condition $\Psi_{z\bar{z}} - \Psi_{\bar{z}z} = 0$, then upon simplifying, it is found that the dependence on λ and $\bar{\lambda}$ factors and the following result must hold

$$\mathbf{F}^-_{\bar{z}} - \mathbf{F}^+_z + [\mathbf{F}^-, \mathbf{F}^+] = 0. \qquad (3.9)$$

Proceeding in an exactly similar manner, the remaining two conditions $\Psi_{zy} - \Psi_{yz} = 0$ and $\Psi_{\bar{z}y} - \Psi_{y\bar{z}} = 0$ give rise to the pair

$$\mathbf{C}_z - \mathbf{F}^-_y + [\mathbf{C}, \mathbf{F}^-] = 0, \qquad \mathbf{C}_{\bar{z}} - \mathbf{F}^+_y + [\mathbf{C}, \mathbf{F}^+] = 0. \qquad (3.10)$$

It remains to demonstrate the correspondence between the results in (3.9) and (3.10) with those in (3.3) explicitly. To this end, replace $\mathbf{F}^{\pm}$ using (3.5) and transform the derivatives using $\partial_z = \partial_x - i\partial_t$ and its conjugate. For example, (3.9) becomes

$$(\mathbf{A}_x - i\mathbf{B}_x + i\mathbf{A}_t - \mathbf{B}_t) - (\mathbf{A}_x + i\mathbf{B}_x - i\mathbf{A}_t + \mathbf{B}_t) + (\mathbf{A} - i\mathbf{B})(\mathbf{A} + i\mathbf{B}) - (\mathbf{A} + i\mathbf{B})(\mathbf{A} - i\mathbf{B}) = 0. \quad (3.11)$$

This reduces to the second equation in (3.3), and the others come out in like fashion.

Based on Theorem 3.3, an associated linear system can be written down which can be regarded as a sequence or hierarchy of linear systems.

Theorem 3.3. An associated hierarchy of the deformed Gauss-Mainardi-Codazzi system (3.3) can be produced by means of the compatibility condition for the following linear system,

$$\Psi_z = \frac{\lambda^2}{1-\lambda^4}(\mathbf{F}^+ - \lambda^2\mathbf{F}^-)\Psi + \frac{1}{1-\lambda^4}(\mathbf{F}^- - \lambda^2\mathbf{F}^+)\Psi,$$
$$\Psi_{\bar{z}} = \frac{\lambda^2}{1-\lambda^4}(\mathbf{F}^- - \lambda^2\mathbf{F}^+)\Psi + \frac{1}{1-\lambda^4}(\mathbf{F}^+ - \lambda^2\mathbf{F}^-)\Psi, \qquad (3.12)$$
$$\Psi_y = -i\frac{\lambda^n}{1-\lambda^4}[\lambda^2(\mathbf{F}^- - \lambda^2\mathbf{F}^+) + \mathbf{F}^+ - \lambda^2\mathbf{F}^-]\Psi + \sum_{k=0}^{m}\lambda^k\mathbf{F}_k\Psi.$$

In (3.12), it suffices to take λ real and determine $\mathbf{F}_k$ order by order.

4 Self-Dual Yang-Mills and the Gauss-Mainardi-Codazzi Equation as an Exact Reduction

The Yang-Mills equations have a natural geometric interpretation (Bracken, 1999). The covariant derivatives can be used to obtain a local representation of a connection on a principle fibre bundle over a manifold M. Let G be the Lie gauge group, LG the Lie algebra, and $\{x^\mu\}_{0,\cdots,3}$ coordinates on M, which may be $\mathbb{R}^4, \mathbb{R}^{1,3}, \mathbb{R}^{2,2}$. For a potential $\mathcal{A}_\mu(x) \in LG$, the covariant derivative is given by $\mathcal{D}_\mu = \partial_\mu - \mathcal{A}_\mu$. The curvature two-form F can be expressed as

$$F = \mathcal{D}\mathcal{A} = d\mathcal{A} - \mathcal{A} \wedge \mathcal{A}. \qquad (4.1)$$

Given the curvature two-form F, the Yang-Mills equations[14] can be written in the form $\mathcal{D} * F = 0$ and there is the associated Bianchi identity $\mathcal{D}F = 0$. Under gauge transformations, $\mathcal{A}_\mu \to g^{-1}\mathcal{A}_\mu g + g^{-1}\partial_\mu g$, $g \in G$, the components of the curvature two-form transform as $F_{\mu\nu} g^{-1} F_{\mu\nu} g$. This corresponds to the transformation of the fibres by the right action of the structure group G on the principle bundle. This procedure will yield the Yang-Mills equations which are a set of coupled second-order partial differential equations in four dimensions for the LG-valued gauge potentials $\mathcal{A}_\mu$. However, any F that satisfies the condition

$$*F = \lambda F \tag{4.2}$$

for some constant λ also satisfies the Yang-Mills equation when F satisfies $\mathcal{D}F = 0$. Applying the Hodge operator to both sides of (4.2), there obtains

$$**F = \lambda * F = \lambda^2 F. \tag{4.3}$$

However, $**F = gF$ where g is the determinant of the metric on the manifold. This establishes the value of λ to be $\lambda = \pm 1$ for $\mathbb{R}^4, \mathbb{R}^{2,2}$ and $\lambda = \pm i$ for $\mathbb{R}^{3,1}$. All solutions to the system of equations $*F = \pm iF$ are trivial. In $\mathbb{R}^4$ with standard metric, it is straightforward to work out $*F$ from the definition.

Let us introduce a new system of coordinates which will match what we seek upon reduction

$$\begin{aligned} \sigma &= x^1 + ix^2 = w + iy, \qquad \tau = x^0 - ix^3 = x + it, \\ \bar{\sigma} &= x^1 - ix^2 = w - iy, \qquad \bar{\tau} = x^0 + ix^3 = x - it. \end{aligned} \tag{4.4}$$

Theorem 4.1. The self-dual Yang-Mills equations take the form

$$F_{\sigma\tau} = 0, \qquad F_{\bar{\sigma}\bar{\tau}} = 0, \qquad F_{\sigma\bar{\sigma}} + F_{\tau\bar{\tau}} = 0. \tag{4.5}$$

in coordinates (4.4). Self-dual system (4.2) results directly from the compatibility conditions of the isospectral linear problem given in the following form,

$$(\partial_\sigma + \lambda\partial_{\bar{\tau}})\Psi = (\mathcal{A}_\sigma + \lambda\mathcal{A}_{\bar{\tau}})\Psi, \qquad (\partial_\tau - \lambda\partial_{\bar{\sigma}})\Psi = (\mathcal{A}_\tau - \lambda\mathcal{A}_{\bar{\sigma}})\Psi. \tag{4.6}$$

In (4.6) λ is the spectral parameter and Ψ is a local section of the Yang-Mills fibre bundle (Bracken, 2005; Chakravarty and Kent, 1995; Ablowitz et al., 2003).

Theorem 4.2. The deformed Gauss-Mainardi-Codazzi system (3.3) can be obtained as a direct result of a particular reduction of the self-dual Yang-Mills system (4.5).

Proof: It is simply required to transform the derivatives between relevant coordinates so that the required correspondence with (3.3) can be recognized, and to state the specific reduction. Thus, the derivatives are related as $\partial_\sigma = \partial_w - i\partial_y$, $\partial_{\bar{\sigma}} = \partial_w + i\partial_y$, $\partial_\tau = \partial_x - i\partial_t$, $\partial_{\bar{\tau}} = \partial_x + i\partial_t$.

The reduction of (4.5) for the components of the gauge potential which will produce the required reduction is given explicitly as

$$\mathcal{A}_\sigma = -i\mathbf{C}, \qquad \mathcal{A}_{\bar{\sigma}} = i\mathbf{C}, \qquad \mathcal{A}_\tau = \mathbf{A} - i\mathbf{B}, \qquad \mathcal{A}_{\bar{\tau}} = \mathbf{A} + i\mathbf{B}. \tag{4.7}$$

The quantities $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ on the right-hand side of (4.7) are taken to be independent of the w variable and, for example, may be selected as elements of $so(3)$.

The first equation in (4.5) then takes the form,

$$-i\partial_y(\mathbf{A}-i\mathbf{B})-(\partial_x-i\partial_t)(-i\mathbf{C})-[-i\mathbf{C},\mathbf{A}-i\mathbf{B}] = -i(\partial_y\mathbf{A}-\partial_x\mathbf{C}-[\mathbf{C},\mathbf{A}])-(\partial_y\mathbf{B}-\partial_t\mathbf{C}-[\mathbf{C},\mathbf{B}]) = 0.$$

The second in (4.5) gives a similar result and addition and subtraction of these generates the two equations with $\mathbf{C}$ in (3.3).

The third equation of (4.5) assuming independence of w gives

$$-i\partial_y(i\mathbf{C}) - i\partial_y(-i\mathbf{C}) - [-i\mathbf{C}, i\mathbf{C}] + (\partial_x - i\partial_t)(\mathbf{A}+i\mathbf{B}) - (\partial_x + i\partial_t)(\mathbf{A}-i\mathbf{B}) - [\mathbf{A}-i\mathbf{B}, \mathbf{A}+i\mathbf{B}] = 0.$$

This simplifies to the third equation $\mathbf{B}_x - \mathbf{A}_t + [\mathbf{B}, \mathbf{A}] = 0$.

5 Some Integrable Systems as Reductions

It is useful at this point to have an idea as to which types of equations can be accounted for as specific reductions of (3.3) or (3.4). It will be shown that some very well-known integrable systems such as the sine-Gordon equation and a generalized Korteweg-de Vries equation can be obtained from what has been developed here.

(i) Let σ_j be the Pauli matrices in standard form with $\sigma^{\pm} = \sigma_1 \pm i\sigma_2$. Let the quantities $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be the 2×2 matrices defined to be

$$\begin{aligned} \mathbf{A} &= \sqrt{2} i\lambda\sigma_3 + \frac{1}{\sqrt{2}}\bar{q}\sigma^+ + \frac{1}{\sqrt{2}} q\sigma^-, \\ \mathbf{B} &= -\frac{i}{2} f\sigma_3 - 3\lambda\bar{q}\sigma^+ - 3\lambda q\sigma^-, \\ \mathbf{C} &= -\frac{i}{\sqrt{2}}\lambda\sigma_3 + \frac{1}{\sqrt{2}}\bar{q}\sigma^+ + \frac{1}{\sqrt{2}} q\sigma^-. \end{aligned} \tag{5.1}$$

In (4.1), λ is a real parameter, q a complex-valued function and to begin with f is an arbitrary function $f = f(q, \bar{q}, \varphi)$. Substituting matrices (4.1) into system (3.3), the following results appear as matrix elements of the resulting matrices,

$$f_x = 0, \quad f_y = 0, \tag{5.2}$$

$$\sqrt{2} q_t + 6\lambda q_x + 12\sqrt{2} i\lambda^2 q - \sqrt{2} iqf = 0, \tag{5.3}$$

$$\sqrt{2} q_t + 6\lambda q_y - 6\sqrt{2} i\lambda^2 q - \sqrt{2} iqf = 0, \tag{5.4}$$

$$\sqrt{2}(q_y - q_x) = 6i\lambda q, \tag{5.5}$$

as well as the conjugates of (5.3)-(5.5). Adding (5.3) and (5.4), we obtain

$$q_t + \frac{3}{\sqrt{2}}\lambda(q_x + q_y) + 3i\lambda^2 q - iqf = 0. \tag{5.6}$$

Differentiating (5.4) with respect to x and y then adding the results, this gives

$$\frac{i}{2}(q_{yy} - q_{xx}) = \frac{3}{\sqrt{2}}\lambda(q_x + q_y). \tag{5.7}$$

Substituting (5.7) into (5.6) produces the result

$$q_t + \frac{i}{2}(q_{xx} - q_{yy}) + i(3\lambda^2 - f)q = 0. \tag{5.8}$$

A Davey-Stewartson type system results from this by selecting f to have the form,

$$f = |q|^2 + \varphi_x - \varphi_y + 3\lambda^2,$$

and this system is given by

$$\begin{aligned} iq_t + \tfrac{1}{2}(q_{yy} - q_{xx}) + (|q|^2 + \varphi_x - \varphi_y)q &= 0, \\ \varphi_{yy} - \varphi_{xx} &= |q|_x^2 + |q|_y^2. \end{aligned} \tag{5.9}$$

(ii) Suppose the last row and column matrix elements of (3.4) are equal to zero, that is take $\tau = \sigma = \omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0$. The matrices collapse to 2×2 form which commute so (3.4) reduce to

$$\kappa_y = \omega_{3x}, \qquad \kappa_t = \gamma_{3x}, \qquad \gamma_{3y} = \omega_{3t}. \tag{5.10}$$

Specific integrable equations can be obtained out of these by choosing the remaining functions in (5.10) in specific ways. Some examples are given.

(a) Define

$$\partial_x^{-1}\,\kappa(x,y) = \int_{-\infty}^{x} ds\,\kappa(s,y).$$

Let $\gamma_3 = \partial_x^{-1}\sin(u(x,t))$ and $\kappa = u_x$, all quantities independent of y and $\omega_3 = 0$. Then the second equation in (5.10) implies that u satisfies

$$u_{xt} = \sin(u). \tag{5.11}$$

(b) Suppose γ_3 is chosen to have the form

$$\gamma_3 = -\kappa_{xx} - \alpha\kappa^m,$$

where α is a real constant. Then (5.10) implies that κ satisfies the following generalized Korteweg-de Vries equation (Bracken, 2007)

$$\kappa_t + \kappa_{xxx} + \alpha(\kappa^m)_x = 0. \tag{5.12}$$

(iii) Define γ_3 to have the form,

$$\gamma_3 = -\kappa_{xx} - 3\kappa^2 - 3\alpha^2\partial_x^{-1}\omega_y.$$

Thus by (5.10), κ satisfies

$$\kappa_t + \kappa_{xxx} + 3(\kappa^2)_x + 3\alpha^2\omega_{3y} = 0. \tag{5.13}$$

By the first equation in (5.10), $\kappa_y = \omega_{3x}$ so

$$\kappa_{yy} = \omega_{3xy}. \tag{5.14}$$

Differentiating both sides of (5.13) with respect to x and substituting (5.14) into the result, we get

$$(\kappa_t + \kappa_{xxx} + 6\kappa\kappa_x)_x + 3\alpha^2\kappa_{yy} = 0. \tag{5.15}$$

This is called the Kadomtsev-Petriashvili equation.

(iv) The Lame system can be developed by taking the matrices in (3.3) to have the form

$$\mathbf{A} = \begin{pmatrix} 0 & -\beta_{21} & -\beta_{31} \\ \beta_{21} & 0 & 0 \\ \beta_{31} & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & \beta_{23} \\ -\beta_{13} & -\beta_{23} & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & \beta_{12} & 0 \\ -\beta_{12} & 0 & -\beta_{32} \\ 0 & \beta_{32} & 0 \end{pmatrix}. \tag{5.16}$$

Substituting the matrices in (5.16) into (3.3), the Lame system of equations is obtained,

$$\begin{gathered} \beta_{21,t} + \beta_{12,x} + \beta_{31}\beta_{32} = 0, \qquad \beta_{31,t} - \beta_{21}\beta_{32} = 0, \qquad \beta_{32,x} - \beta_{31}\beta_{12} = 0, \\ \beta_{21,y} - \beta_{31}\beta_{23} = 0, \qquad \beta_{31,y} + \beta_{13,x} + \beta_{21}\beta_{23} = 0 \qquad \beta_{23,x} - \beta_{21}\beta_{13} = 0, \\ \beta_{12,y} - \beta_{32}\beta_{13} = 0, \qquad \beta_{13,t} - \beta_{12}\beta_{23} = 0, \qquad \beta_{32,y} - \beta_{23,t} + \beta_{12}\beta_{13} = 0. \end{gathered} \tag{5.17}$$

This system has been studied previously (Zakharov, 1998).

6 Conclusions

A remarkable relationship between the Gauss-Weingarten system and a linear system has been presented and clarified. It has been shown how this linear system can be deformed to describe deformations of surfaces by introducing an evolution parameter. This has made possible a unified and consistent picture which consists of integrable systems described as certain types of reductions of (3.4) and associated surfaces. Many important systems fall under this category. Thus of special importance is to state the deformed Gauss-Mainardi-Codazzi system (3.3) has been obtained as a particular reduction of a self-dual Yang-Mills system.

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