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# Gaussian Generalized Woodall Numbers

**Orhan Eren [a] and Yüksel Soykan [a*]**

[a] *Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.*

***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors have read and approved the final version of the manuscript.*

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## ABSTRACT

In this work, we define and investigate Gaussian generalized Woodall numbers in detail, and focus on four specific cases: Gaussian modified Woodall numbers, Gaussian modified Cullen numbers, Gaussian Woodall numbers, and Gaussian Cullen numbers. We present some identities and matrices related to these sequences, as well as recurrence relations, Binet's formulas, generating functions, Simpson's formulas, and summation formulas.

*Keywords: Woodall numbers; Cullen numbers; Gaussian Woodall numbers; Gaussian Cullen numbers; Gaussian generalized Woodall numbers; Gaussian modified Woodall numbers; Gaussian modified Cullen numbers.*

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## 1 INTRODUCTION

First, we recall some properties of generalized Woodall numbers. The generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ is defined by the third-order recurrence relation as

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{1.1}$$

**Corresponding author: E-mail: yuksel_soykan@hotmail.com;*

with the initial values $W_0, W_1, W_2$ not all being zero.

A generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (1.1) holds for all integer $n$. For more details, see [48].

Next, we give Binet formula of generalized Woodall numbers.

**Theorem 1.1.** *[[48], Theorem 1.1] Binet formula of generalized Woodall numbers can be given as*

$$W_n = (A_1 + A_2 n) \times 2^n + A_3$$

*where $A_1$, $A_2$ and $A_3$ are defined by*

$$\begin{array}{rcl} A_1 &=& -W_2 + 4W_1 - 3W_0, \\ A_2 &=& \dfrac{W_2 - 3W_1 + 2W_0}{2}, \\ A_3 &=& W_2 - 4W_1 + 4W_0, \end{array}$$

*that is,*

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2}n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{1.2}$$

*Here, $\alpha, \beta$ and $\gamma$ are the roots of the cubic equation*

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2(x-1) = 0,$$

*where*

$$\begin{array}{rcl} \alpha &=& \beta = 2, \\ \gamma &=& 1. \end{array}$$

Now, the first few generalized Woodall numbers with positive subscript and negative subscript are given in the following table.

**Table 1. The first few generalized Woodall numbers with positive subscript and negative subscript**

| $n$ | $W_n$ | $W_{-n}$ |
|---|---|---|
| 0 | $W_0$ | $W_0$ |
| 1 | $W_1$ | $\frac{1}{4}(8W_0 - 5W_1 + W_2)$ |
| 2 | $W_2$ | $\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$ |
| 3 | $4W_0 - 8W_1 + 5W_2$ | $\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$ |
| 4 | $20W_0 - 36W_1 + 17W_2$ | $\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$ |
| 5 | $68W_0 - 116W_1 + 49W_2$ | $\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$ |
| 6 | $196W_0 - 324W_1 + 129W_2$ | $\frac{1}{64}(247W_0 - 243W_1 + 60W_2)$ |

Now, we define four specific cases of the sequence $\{W_n\}$.

The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

$$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, \ldots$$

(sequence A003261 in the OEIS 43). Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [12] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.

The Cullen numbers $\{C_n\}$ are numbers of the form

$$C_n = n \times 2^n + 1.$$

The first few Cullen numbers are:

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, ...$$

(sequence A002064 in the OEIS).

Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [7],[8],[12],[21],[23],[25],[28],[33],[34],[35],[39] and references therein.

Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

$$\begin{aligned} R_n &= 4R_{n-1} - 4R_{n-2} - 1, \\ C_n &= 4C_{n-1} - 4C_{n-2} + 1. \end{aligned}$$

Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7, \tag{1.3}$$

$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \tag{1.4}$$

Modified Woodall sequence $\{G_n\}_{n\geq 0}$ and modified Cullen sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations,

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \quad G_0 = 0, G_1 = 1, G_2 = 5, \tag{1.5}$$

$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \quad H_0 = 3, H_1 = 5, H_2 = 9. \tag{1.6}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining,

$$\begin{aligned} G_{-n} &= 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)}, \\ H_{-n} &= 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)}, \\ R_{-n} &= 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)}, \\ C_{-n} &= 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3),(1.4), (1.5) and (1.6) hold for all integer $n$.

For all integers $n$, modified Woodall, modified Cullen, Woodall and Cullen numbers (using initial conditions in (1.2)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= (n-1)\,2^n + 1, \\ H_n &= 2^{n+1} + 1, \\ R_n &= n \times 2^n - 1, \\ C_n &= n \times 2^n + 1, \end{aligned}$$

respectively.

Now we give some information about Gaussian sequence from literature.

- First we give Gaussian numbers with second order recurreance

  - Horadam [26] introduced Gussian Fibonacci numbers as

    $$GF_n = F_n + iF_{n-1}$$

    where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact,he defined these numbers as $GF_n = F_n + iF_{n-1}$ and he called these numbers as complex Fibonacci numbers.)

  - Pethe and Horadam [36] introduced generalized Gaussian Fibonacci numbers

    $$GF_n = F_n + iF_{n-1}$$

    where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

  - Halıcı and Öz [24] studied Gaussian Pell and Pell Lucas numbers by written, respectively,

    $$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1}, \end{aligned}$$

    where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

  - Aşçı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

    $$\begin{aligned} GJ_n &= J_n + iJ_{n-1}, \\ Gj_n &= j_n + j_{n-1}, \end{aligned}$$

    where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

  - Taşcı [68] introduced and studied Gaussian Mersenne numbers and define by

    $$GM_n = M_n + iM_{n-1}$$

    where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

  - Taşçı [66] introduced and studied Gaussian balancing and Lucas Balancing numbers and given by, respectively,

    $$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + C_{n-1}, \end{aligned}$$

    where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6C_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

  - Ertaş and Yılmaz [2] studied Gaussian Oresme numbers and given by

    $$GT_n = T_n + iT_{n-1}$$

    where $T_n = T_{n-1} - \frac{1}{4}T_{n-2}$, $T_0 = 0$, $T_1 = \frac{1}{2}$.

- Now, we present Gaussian numbers with third order reccurance relations.

  - Soykan, Taşdemir, Okumuş and Göcen [45] presented Gaussian generalized Tribonacci numbers by given

    $$GW_n = W_n + iW_{n-1}$$

    where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition $W_0, W_1, W_2$.

– Taşcı [67] studied Gaussian Padovan and Gaussian Pell- Padovan numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GR_n &= R_n + iR_{n-1}, \end{aligned}$$

where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

– Cerda-Morales [9] defined Gaussian third-order Jacobsthal numbers by given

$$GJ_n = J_n + iJ_{n-1}$$

where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

## 2 GAUSSIAN GENERALIZED WOODALL NUMBERS

In this chapter, we define Gaussian generalized Woodall numbers and we give some properties. Gaussian generalized Woodall numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 5GW_{n-1} - 8GW_{n-2} + 4GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$GW_0 = W_0 + i(\frac{1}{4}(8W_0 - 5W_1 + W_2)),\ GW_1 = W_1 + iW_0,\ GW_2 = W_2 + iW_1,$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = 2GW_{-(n-1)} - \frac{5}{4}GW_{-(n-2)} + \frac{1}{4}GW_{-(n-3)}$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (2.1) hold for all integer $n$. Note that for $n \geq 0$, we get

$$GW_n = W_n + iW_{n-1} \tag{2.2}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}. \tag{2.3}$$

The first few generalized Gaussian Woodall numbers with positive subscript and negative subscript are given in the following table.

**Table 2. The first few generalized Gaussian Woodall numbers**

| $n$ | $GW_n$ | $GW_{-n}$ |
|---|---|---|
| 0 | $W_0 + i\frac{1}{4}(8W_0 - 5W_1 + W_2)$ | $W_0 + i\frac{1}{4}(8W_0 - 5W_1 + W_2)$ |
| 1 | $W_1 + iW_0$ | $\frac{1}{4}(8W_0 - 5W_1 + W_2) + i\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$ |
| 2 | $W_2 + iW_1$ | $\frac{1}{4}(11W_0 - 9W_1 + 2W_2) + i\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$ |
| 3 | $4W_0 - 8W_1 + 5W_2 + iW_2$ | $\frac{1}{16}(52W_0 - 47W_1 + 11W_2) + i\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$ |

We consider four special cases of $GW_n$ :

$GW_n(0, 1, 5 + i) = GG_n$ is the sequence of Gaussian Modified Woodall numbers,
$GW_n(3 + 2i, 5 + 3i, 9 + 5i) = GH_n$ is the sequence of Gaussian Modified Cullen numbers,
$GW_n(-1 - \frac{3}{2}i, 1 - i, 7 + i) = GR_n$ is the sequence of Gaussian Woodall numbers and
$GW_n(1 + \frac{1}{2}i, 3 + i, 9 + 3i) = GC_n$ is the sequence of Gaussian Cullen numbers.

We formally define them as follows. Four special cases of $GW_n$ with the initial conditions are defined by

$$\begin{aligned} GG_n &= 5GG_{n-1} - 8GG_{n-2} + 4GG_{n-3}, \quad GG_0 = 0, GG_1 = 1, GG_2 = 5 + i, \\ GH_n &= 5GH_{n-1} - 8GH_{n-2} + 4GH_{n-3}, \quad GH_0 = 3 + 2i, GH_1 = 5 + 3i, GH_2 = 9 + 5i, \\ GR_n &= 5GR_{n-1} - 8GR_{n-2} + 4GR_{n-3}, \quad GR_0 = -1 - \frac{3}{2}i, GR_1 = 1 - i, GR_2 = 7 + i, \\ GC_n &= 5GC_{n-1} - 8GC_{n-2} + 4GC_{n-3}, \quad GC_0 = 1 + \frac{1}{2}i, GC_1 = 3 + i, GC_2 = 9 + 3i. \end{aligned}$$

Note that for all integers $n$, we obtain

$$\begin{aligned} GG_n &= G_n + iG_{n-1}, \\ GH_n &= H_n + iH_{n-1}, \\ GR_n &= R_n + iR_{n-1}, \\ GC_n &= C_n + iC_{n-1}. \end{aligned}$$

The first few values of Gaussian Modified Woodall numbers, Gaussian Modified Cullen numbers, Gaussian Woodall numbers and Gaussian Cullen numbers with positive and negative subscript are given in the following table.

**Table 3. The first few values of special cases of generalized Gaussian Woodall numbers**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $GG_n$ | $0$ | $1$ | $5+i$ | $17+5i$ | $49+17i$ | $129+49i$ | $321+129i$ | $769+321i$ |
| $GG_{-n}$ | $0$ | $\frac{1}{4}i$ | $\frac{1}{4}+\frac{1}{2}i$ | $\frac{1}{2}+\frac{11}{16}i$ | $\frac{11}{16}+\frac{13}{16}i$ | $\frac{13}{16}+\frac{57}{64}i$ | $\frac{57}{64}+\frac{15}{16}i$ | $\frac{15}{16}+\frac{247}{256}i$ |
| $GH_n$ | $3+2i$ | $5+3i$ | $9+5i$ | $17+9i$ | $33+17i$ | $65+33i$ | $129+65i$ | $257+129i$ |
| $GH_{-n}$ | $3+2i$ | $2+\frac{3}{2}i$ | $\frac{3}{2}+\frac{5}{4}i$ | $\frac{5}{4}+\frac{9}{8}i$ | $\frac{9}{8}+\frac{17}{16}i$ | $\frac{17}{16}+\frac{33}{32}i$ | $\frac{33}{32}+\frac{65}{64}i$ | $\frac{65}{64}+\frac{129}{128}i$ |
| $GR_n$ | $-1-\frac{3}{2}i$ | $1-i$ | $7+i$ | $23+7i$ | $63+23i$ | $159+63i$ | $383+159i$ | $895+383i$ |
| $GR_{-n}$ | $-1-\frac{3}{2}i$ | $-\frac{3}{2}-\frac{3}{2}i$ | $-\frac{3}{2}-\frac{11}{8}i$ | $\frac{11}{8}-\frac{5}{4}i$ | $-\frac{5}{4}-\frac{37}{32}i$ | $-\frac{37}{32}-\frac{35}{32}i$ | $-\frac{35}{32}-\frac{135}{128}i$ | $\frac{135}{128}-\frac{33}{32}i$ |
| $GC_n$ | $1+\frac{1}{2}i$ | $3+i$ | $9+3i$ | $25+9i$ | $65+25i$ | $161+65i$ | $385+161i$ | $897+385i$ |
| $GC_{-n}$ | $1+\frac{1}{2}i$ | $\frac{1}{2}+\frac{1}{2}i$ | $\frac{1}{2}+\frac{5}{8}i$ | $\frac{5}{8}+\frac{3}{4}i$ | $\frac{3}{4}+\frac{27}{32}i$ | $\frac{27}{32}+\frac{29}{32}i$ | $\frac{29}{32}+\frac{121}{128}i$ | $\frac{121}{128}+\frac{31}{32}i$ |

We now present the Binet formula for the Gaussian generalized Woodall numbers.

**Theorem 2.1.** *The Binet's formula for the Gaussian generalized Woodall numbers is* $GW_n = (((-W_2 + 4W_1 - 3W_0) + \frac{W_2-3W_1+2W_0}{2}n)2^n + (W_2 - 4W_1 + 4W_0)) + i(((-W_2 + 4W_1 - 3W_0) + \frac{W_2-3W_1+2W_0}{2}(n-1))2^{n-1} + (W_2 - 4W_1 + 4W_0))$.

*Proof.* The proof follows from (1.2) and (2.2). □

The previous Theorem gives the following results, as special cases.

**Corollary 2.2.** *For all $n$ we have the following Binet's Formulas*

**(a)** $GG_n = i2^{n-1}(n-2) + 2^n(n-1) + 1 + i.$

**(b)** $GH_n = 2i2^{n-1} + 2 \times 2^n + 1 + i.$

**(c)** $GR_n = i2^{n-1}(n-1) + 2^n n - 1 - i.$

**(d)** $GC_n = i2^{n-1}(n-1) + 2^n n + 1 + i.$

The following Theorem presents the generating function of Gaussian generalized Woodall numbers.

**Theorem 2.3.** *The generating function of Gaussian generalized Woodall numbers is given as*

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0 + (GW_1 - 5GW_0)x + (GW_2 - 5GW_1 + 8GW_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{2.4}$$

*Proof.* Let

$$f_{GW_n}(x)=\sum_{n=0}^{\infty}GW_nx^n$$

be generating function of Gaussian generalized Woodall numbers. Then using the definition of Gaussian Woodall numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain (note the shift in the index $n$ in the third line)

$$\begin{aligned}
(1-5x+8x^2-4x^3)f_{GW_n}(x) &= \sum_{n=0}^{\infty}GW_nx^n-5x\sum_{n=0}^{\infty}GW_nx^n+8x^2\sum_{n=0}^{\infty}GW_nx^n-4x^3\sum_{n=0}^{\infty}GW_nx^n,\\
&= \sum_{n=0}^{\infty}GW_nx^n-5\sum_{n=0}^{\infty}GW_nx^{n+1}+8\sum_{n=0}^{\infty}GW_nx^{n+2}-4\sum_{n=0}^{\infty}GW_nx^{n+3},\\
&= \sum_{n=0}^{\infty}GW_nx^n-5\sum_{n=1}^{\infty}GW_{n-1}x^n+8\sum_{n=2}^{\infty}GW_{n-2}x^n-4\sum_{n=3}^{\infty}GW_{n-3}x^n,\\
&= (GW_0+GW_1x+GW_2x^2)-5(GW_0x+GW_1x^2)+8GW_0x^2\\
&\quad +\sum_{n=3}^{\infty}(GW_n-5GW_{n-1}+8GW_{n-2}-4GW_{n-3})x^n,\\
&= GW_0+GW_1x+GW_2x^2-5GW_0x-5GW_1x^2+8GW_0x^2,\\
&= GW_0+(GW_1-5GW_0)x+(GW_2-5GW_1+8GW_0)x^2.
\end{aligned}$$

Now, it follows that

$$f_{GW_n}(x)=\frac{GW_0+(GW_1-5GW_0)x+(GW_2-5GW_1+8GW_0)x^2}{1-5x+8x^2-4x^3}.$$

This completes the proof. □

The previous Theorem gives the following results as particular examples:

$$f_{GG_n}(x)=\frac{x+ix^2}{1-5x+8x^2-4x^3}, \tag{2.5}$$

$$f_{GH_n}(x)=\frac{(8+6i)\,x^2-(10+7i)\,x+3+2i}{1-5x+8x^2-4x^3}, \tag{2.6}$$

$$f_{GR_n}(x)=\frac{-(6+6i)\,x^2+(6+\frac{13}{2}i)x-1-\frac{3}{2}i}{1-5x+8x^2-4x^3}, \tag{2.7}$$

$$f_{GC_n}(x)=\frac{(2+2i)\,x^2-(2+\frac{3}{2}i)x+1+\frac{1}{2}i}{1-5x+8x^2-4x^3}. \tag{2.8}$$

# 3 SOME IDENTITIES RELATED TO GAUSSIAN MODIFIED WOODALL, GAUSSIAN MODIFIED CULLEN, GAUSSIAN WOODALL AND GAUSSIAN CULLEN NUMBERS

In this section, we obtain some identities on Gaussian modified Woodall, Gaussian modified Cullen, Gaussian Woodall and Gaussian Cullen numbers.

**Theorem 3.1.** *The following equations hold for all integer* $n$.

$$GH_n = 2GG_{n+2} - 7GG_{n+1} + 6GG_n, \tag{3.1}$$

$$GH_n = 3GG_{n+1} - 10GG_n + 8GG_{n-1}, \tag{3.2}$$

$$GR_n = -2GC_{n+2} + 8GC_{n+1} - 7GC_n,$$

$$GG_n = -\frac{1}{2}GC_{n+2} + \frac{3}{2}GC_{n+1}, \tag{3.3}$$

$$GC_n = -\frac{7}{4}GR_{n+3} + \frac{27}{4}GR_{n+2} - 6GR_{n+1}, \tag{3.4}$$

$$GH_n = -\frac{1}{2}GR_{n+3} + \frac{5}{2}GR_{n+2} - 3GR_{n+1}, \tag{3.5}$$

$$GH_n = 5GG_n - 16GG_{n-1} + 12GG_{n-2}. \tag{3.6}$$

Proof. To proof identity (3.1), we can write

$$GH_n = aGG_{n+2} + bGG_{n+1} + cGG_n$$

and solving the system of equations

$$\begin{aligned} GH_0 &= aGG_2 + bGG_1 + cGG_0, \\ GH_1 &= aGG_3 + bGG_2 + cGG_1, \\ GH_2 &= aGG_4 + bGG_3 + cGG_2. \end{aligned}$$

We find that $a = 2$, $b = -7$, $c = 6$. Or using the relations $GH_n = H_n + iH_{n-1}$, $GG_n = G_n + iG_{n-1}$ and identity $H_n = 2G_{n+2} - 7G_{n+1} + 6G_n$, we obtain the identity (3.1). The others can be found similarly. □

**Lemma 3.2.** *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are given as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n}x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

*and*

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1}x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

*respectively.*

The next Theorem presents the generating functions of even and odd-indexed Gaussian generalized Woodall sequences.

**Theorem 3.3.** *The generating functions of the sequences* $GW_{2n}$ *and* $GW_{2n+1}$ *are given by*

$$f_{GW_{2n}}(x) = \frac{GW_0 - (9GW_0 - GW_2)x + (44GW_0 - 36GW_1 + 8GW_2)x^2}{1 - 9x + 24x^2 - 16x^3}$$

*and*

$$f_{GW_{2n+1}}(x) = \frac{GW_1 + (4GW_0 - 17GW_1 + 5GW_2)x + (32GW_0 - 20GW_1 + 4GW_2)x^2}{1 - 9x + 24x^2 - 16x^3}$$

*respectively.*

*Proof.* Both statements are consequences of Lemma (3.2) applied to (2.4) and some lengthy algebraic calculations. □

The previous theorem gives the following corollaries as particular examples.

**Corollary 3.4.** *We have the followings:*

**(a)** $f_{GR_{2n}}(x) = -\frac{(24+22i)x^2-(16+\frac{29}{2}i)x+1+\frac{3}{2}i}{1-9x+24x^2-16x^3}$ *and* $f_{GR_{2n+1}} = (x)\frac{-(24+24i)x^2+(14+16i)x+1-i}{1-9x+24x^2-16x^3}$.

**(b)** $f_{GC_{2n}}(x) = \frac{(8+10i)x^2-\frac{3}{2}ix+1+\frac{1}{2}i}{1-9x+24x^2-16x^3}$ *and* $f_{GC_{2n+1}}(x) = \frac{(8+8i)x^2-2x+3+i}{1-9x+24x^2-16x^3}$.

**(c)** $f_{GG_{2n}}(x) = \frac{(4+8i)x^2+(5+i)x}{1-9x+24x^2-16x^3}$ *and* $f_{GG_{2n+1}}(x) = \frac{4ix^2+(8+5i)x+1}{1-9x+24x^2-16x^3}$.

**(d)** $f_{GH_{2n}}(x) = \frac{(24+20i)x^2-(18+13i)x+3+2i}{1-9x+24x^2-16x^3}$ *and* $f_{GH_{2n+1}}(x) = \frac{(32+24i)x^2-(28+18i)x+5+3i}{1-9x+24x^2-16x^3}$.

From Corollary (3.4) we can obtain the following corollary which presents the identities on Gaussian Woodall sequences.

**Corollary 3.5.** *We have the following identities:*

**(a)** $(4+8i)\,GH_{2n-4} + (5+i)\,GH_{2n-2} = (24+20i)GG_{2n-4} - (18+13i)\,GG_{2n-2} + (3+2i)GG_{2n}$.

**(b)** $(4+8i)\,GH_{2n-3} + (5+i)\,GH_{2n-1} = (32+24i)\,GG_{2n-4} - (28+18i)\,GG_{2n-2} + (5+3i)GG_{2n}$.

**(c)** $-(24+24i)\,GG_{2n-4} + (14+16i)\,GG_{2n-2} + (1-i)GG_{2n} = (4+8i)GR_{2n-3} + (5+i)\,GR_{2n-1}$.

**(d)** $-(24+24i)\,GG_{2n-3} + (14+16i)\,GG_{2n-1} + (1-i)GG_{2n+1} = 4iGR_{2n-3} + (8+5i)\,GR_{2n-1} + GR_{2n+1}$.

**(e)** $(8+10i)\,GG_{2n-4} - \frac{3}{2}iGG_{2n-2} + (1+\frac{1}{2}i)GG_{2n} = (4+8i)GC_{2n-4} + (5+i)\,GC_{2n-2}$.

**(f)** $(8+10i)\,GG_{2n-3} - \frac{3}{2}iGG_{2n-1} + (1+\frac{1}{2}i)GG_{2n+1} = 4iGC_{2n-4} + (8+5i)\,GC_{2n-2} + GC_{2n}$.

**(g)** $(8+8i)\,GG_{2n-4} - 2GG_{2n-2} + (3+i)GG_{2n} = (4+8i)GC_{2n-3} + (5+i)\,GC_{2n-1}$.

**(h)** $(8+8i)GG_{2n-3} - 2GG_{2n-1} + (3+i)GG_{2n+1} = 4iGC_{2n-3} + (8+5i)\,GC_{2n-1} + GC_{2n+1}$.

**(i)** $-(24+22i)GG_{2n-4} + (16+\frac{29}{2}i)GG_{2n-2} - (1+\frac{3}{2}i)GG_{2n} = (4+8iGR_{2n-4} + (5+i)\,GR_{2n-2}$.

**(j)** $-(24+22i)GG_{2n-3} + (16+\frac{29}{2}i)GG_{2n-1} - (1+\frac{3}{2}i)GG_{2n+1} = 4iGR_{2n-4} + (8+5i)\,GR_{2n-2} + GR_{2n}$.

**(k)** $4iGH_{2n-4} + (8+5i)\,GH_{2n-2} + GH_{2n} = (24+20i)\,GG_{2n-3} - (18+13i)\,GG_{2n-1} + (3+2i)GG_{2n+1}$.

**(l)** $4iGH_{2n-3} + (8+5i)\,GH_{2n-1} + GH_{2n+1} = (32+24i)\,GG_{2n-3} - (28+18i)\,GG_{2n-1} + (5+3i)GG_{2n+1}$.

**(m)** $-(24+22i)C(2n-3) + \left(16+\frac{29}{2}i\right)C(2n-1) - (1+\frac{3}{2}i)C(2n+1) = (8+8i)\,R(2n-4) - 2R(2n-2) + (3+i)R(2n)$.

**(n)** $-(24+24i)\,C(2n-4) + (14+16i)\,C(2n-2) + (1-i)C(2n) = (8+10i)\,R(2n-3) - \frac{3}{2}iR(2n-1) + (1+\frac{1}{2}i)R(2n+1)$.

*Proof.* From (3.4) we obtain

$$((4+8i)\,x^2 + (5+i)\,x)f_{GH_{2n}} = ((24+20i)\,x^2 - (18+13i)\,x + 3 + 2i)f_{GG_{2n}}.$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ((5+i)\,x + (4+8i)\,x^2)\sum_{n=0}^{\infty} GH_{2n}x^n \\
&= (5+i)\,x\sum_{n=0}^{\infty} GH_{2n}x^n + (4+8i)\,x^2\sum_{n=0}^{\infty} GH_{2n}x^n \\
&= (5+i)\sum_{n=0}^{\infty} GH_{2n}x^{n+1} + (4+8i)\sum_{n=0}^{\infty} GH_{2n}x^{n+2} \\
&= (5+i)\sum_{n=1}^{\infty} GH_{2n-2}x^n + (4+8i)\sum_{n=2}^{\infty} GH_{2n-4}x^n \\
&= (5+i)\,GH_0x\sum_{n=2}^{\infty} GH_{2n-2}x^n + (4+8i)\sum_{n=2}^{\infty} GH_{2n-4}x^n \\
&= (5+i)\,(3+2i)x + \sum_{n=2}^{\infty}((4+8i)\,GH_{2n-4} + (5+i)\,GH_{2n-2})\,x^n
\end{aligned}$$

whereas the RHS is

$$\begin{aligned} RHS &= (3+2i-(18+13i)\,x+(24+20i)\,x^2)\sum_{n=0}^{\infty} GG_{2n}x^n \\ &= (3+2i)\sum_{n=0}^{\infty} GG_{2n}x^n-(18+13i)\,x\sum_{n=0}^{\infty} GG_{2n}x^n+(24+20i)\,x^2\sum_{n=0}^{\infty} GG_{2n}x^n \\ &= (3+2i)\sum_{n=0}^{\infty} GG_{2n}x^n-(18+13i)\sum_{n=0}^{\infty} GG_{2n}x^{n+1}+(24+20i)\sum_{n=0}^{\infty} GG_{2n}x^{n+2} \\ &= (3+2i)\sum_{n=0}^{\infty} GG_{2n}x^n-(18+13i)\sum_{n=1}^{\infty} GG_{2n-2}x^n+(24+20i)\sum_{n=2}^{\infty} GG_{2n-4}x^n \\ &= (3+2i)(GG_0+GG_2x)\sum_{n=2}^{\infty} GG_{2n}x^n-(18+13i)\,(GG_0x)\sum_{n=2}^{\infty} GG_{2n-2}x^n \\ &\quad +(24+20i)\sum_{n=2}^{\infty} GG_{2n-4}x^n \\ &= (3+2i)\,(5+i)\,x+\sum_{n=2}^{\infty}((24+20i)GG_{2n-4}-(18+13i)\,GG_{2n-2}+(3+2i)GG_{2n})\,x^n. \end{aligned}$$

Compare the coefficients and the proof of the first identity (a) is done. The other identities can be proved similarly. □

We present an identity related with Gaussian general Woodall numbers and Woodall numbers.

**Theorem 3.6.** *For all $n,m\in\mathbb{Z}$, the following identity holds:*

$$GW_{m+n}=G_{m+1}GW_n+(-8G_m+4G_{m-1})GW_{n-1}+4G_mGW_{n-2}. \tag{3.7}$$

*Proof.* First, we assume that $m,n\geq 0$. The other cases can be proved similarly. We prove the identity (3.7) by induction on $m$. If $m=0$ then

$$GW_n=G_1GW_n+(-8G_0+4G_{-1})GW_{n-1}+4G_0GW_{n-2}$$

which is true because $G_{-1}=0$, $G_0=0$, $G_1=1$. Assume that the equaliy holds for $m\leq k$. For $m=k+1$, we have

$$\begin{aligned} GW_{(k+1)+n} &= 5GW_{n+k}-8GW_{n+k-1}+4GW_{n+k-2} \\ &= 5(G_{k+1}GW_n+(-8G_k+4G_{k-1})GW_{n-1}+4G_kGW_{n-2}) \\ &\quad -8(G_kGW_n+(-8G_{k-1}+4G_{k-2})GW_{n-1}+4G_{k-1}GW_{n-2}) \\ &\quad +4(G_{k-1}GW_n+(-8G_{k-2}+4G_{k-3})GW_{n-1}+4G_{k-2}GW_{n-2}) \\ &= (5G_{k+1}-8G_k+4G_{k-1})GW_n+(-8(G_k+G_{k-1}+G_{k-2}) \\ &\quad +4(G_{k-1}+G_{k-2}+G_{k-3}))GW_{n-1}+4(G_k+G_{k-1}+G_{k-2})GW_{n-2} \\ &= G_{k+2}GW_n+(-8G_{k+1}+4G_k)GW_{n-1}+4G_{k+1}GW_{n-2} \\ &= G_{(k+1)+1}GW_n+(-8G_{k+1}+4G_{(k+1)-1})GW_{n-1}+4G_{k+1}GW_{n-2}. \end{aligned}$$

By mathematical induction on $m$, this proves (3.6). □

The previous Theorem gives the following results as particular examples:

For all $n, m \in \mathbb{Z}$, we have ( taking $GW_n = GG_n$ or $GW_n = GH_n$ or $GW_n = GR_n$ or $GW_n = GC_n$ )

$$\begin{aligned} GG_{m+n} &= G_{m+1}GG_n + (-8G_m + 4G_{m-1})GG_{n-1} + 4G_mGG_{n-2}, \\ GH_{m+n} &= G_{m+1}GH_n + (-8G_m + 4G_{m-1})GH_{n-1} + 4G_mGH_{n-2}, \\ GR_{m+n} &= G_{m+1}GR_n + (-8G_m + 4G_{m-1})GR_{n-1} + 4G_mGR_{n-2}, \\ GC_{m+n} &= G_{m+1}GC_n + (-8G_m + 4G_{m-1})GC_{n-1} + 4G_mGC_{n-2}. \end{aligned}$$

# 4 SIMPSON'S FORMULA

In this chapter, we present Simpson's formula of generalized Gaussian Woodall numbers.

**Theorem 4.1.** *(Simpson's formula of generalized Gaussian Woodall numbers). For all integers $n$, we have*

$$\begin{aligned} \begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} &= 4^n \begin{vmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{vmatrix} \\ &= 4^n(\frac{1}{800} + \frac{7}{800}i)(W_0 - W_1 + \frac{1}{4}W_2)((2-14i)W_0 - (3-21i)W_1 \\ &\quad +(1-7i)W_2)^2. \end{aligned}$$

Proof. Use [[46],Theorem 3.1]. □

From the Theorem (4.1) we get the following corollary.

**Corollary 4.2.** *For all integer n, we get the following identities.*

**(a)** $\begin{vmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{vmatrix} = (1-7i)\, 2^{2n-4}.$

**(b)** $\begin{vmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{vmatrix} = 0.$

**(c)** $\begin{vmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{vmatrix} = -(1-7i)\, 2^{2n-4}.$

**(d)** $\begin{vmatrix} GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+1} & GC_n & GC_{n-1} \\ GC_n & GC_{n-1} & GC_{n-2} \end{vmatrix} = (1-7i)\, 2^{2n-4}.$

# 5 SUM FORMULAS

In this chapter, we give some sum formulas of generalized Gaussian Woodall numbers.

**Theorem 5.1.** *For all integers $n \geq 0$, we have the following formulas:*

**(a)** $\sum_{k=0}^{n} GW_k = \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9) + i(\frac{1}{4}(28 + 16n - 5 \times 2^{n+2} + 2^{n+2}n)W_0 + (-33 - 16n + 7 \times 2^{n+2} - 3 \times 2^{n+1}n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n)W_2).$

**(b)** $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64) + i(\frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32))$.

**(c)** $\sum_{k=0}^{n} GW_{2k} = \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32) + i((\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) + \frac{1}{18}W_2((18n - 2^{2n+1}(2n-1) + 2^{2n+3}(2n-3) + \frac{53}{2}) - \frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2}))$.

*Proof.*

**(a)** When we use (2.2),

$$\sum_{k=0}^{n} GW_k = \sum_{k=0}^{n} W_k + i\sum_{k=0}^{n} W_{k-1}.$$

So, then we obtain

$$\sum_{k=0}^{n} W_k = \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9))$$

and

$$\sum_{k=0}^{n} W_{k-1} = (\frac{1}{4}((28 + 16n - 5 \times 2^{n+2} + 2^{n+2}n)W_0 + (-33 - 16n + 7 \times 2^{n+2} - 3 \times 2^{n+1}n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n)W_2))$$

from sum formulas on the Generalized Woodall Sequence article. We get

$$\sum_{k=0}^{n} GW_k = \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2) + 6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8) + 22) + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3) + 9)) + i(\frac{1}{4}((28 + 16n - 5 \times 2^{n+2} + 2^{n+2}n)W_0 + (-33 - 16n + 7 \times 2^{n+2} - 3 \times 2^{n+1}n)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n)W_2)).$$

**(b)** When we use (2.1), we obtain the following equalities: If we rearrange the above equalities, we obtain. Now, if we add the above equations by side by, we get

$$\sum_{k=0}^{n} GW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + i\sum_{k=0}^{n} W_{2k}$$

and so we know

$$\sum_{k=0}^{n} W_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64)$$

and

$$\sum_{k=0}^{n} W_{2k} = ((\frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32)).$$

We get

$$\begin{aligned}\sum_{k=0}^{n} GW_{2k+1} &= \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) \\ &+2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64) \\ &+i((\frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) \\ &+2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32)).\end{aligned}$$

**(c)** We know

$$\begin{aligned}\sum_{k=0}^{n} W_{2k} &= ((\frac{1}{9}W_0(36n - 2^{2n+2}((2n-1) + 2^{2n+4}((2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) \\ &+2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32))\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} W_{2k-1} &= (\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) + \frac{1}{18}W_2(18n - 2^{2n+1}(2n-1) \\ &+2^{2n+3}(2n-3) + \frac{53}{2}) - \frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2})).\end{aligned}$$

So we know

$$\sum_{k=0}^{n} GW_{2k} = \sum_{k=0}^{n} W_{2k} + i\sum_{k=0}^{n} W_{2k-1}.$$

We get

$$\begin{aligned}\sum_{k=0}^{n} GW_{2k} &= \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) \\ &+2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32) \\ &+i(\frac{1}{9}W_0(36n - 2^{2n+1}(2n-2) + 2^{2n+3}(2n-4) + 46) + \frac{1}{18}W_2((18n - 2^{2n+1}(2n-1) \\ &+2^{2n+3}(2n-3) + \frac{53}{2}) - \frac{1}{18}W_1(72n - 2^{2n+1}(6n-5) + 2^{2n+3}(6n-11) + \frac{201}{2})).\end{aligned}$$

This completes the proof. □

As special cases of above Theorem, we have the following four Corollary, we get the following corollary:
First, taking $GW_n = GG_n$ with $GG_0 = 0, GG_1 = 1, GG_2 = 5 + i$.

**Corollary 5.2.** *(Sum of the Gaussian modified Woodall numbers). For* $n \geq 0$ *we have the following formulas:*

**(a)** $\sum_{k=0}^{n} GG_k = (1+i)n + (2+i)2^n n - (4+3i)2^n + (4+3i)$.

**(b)** $\sum_{k=0}^{n} GG_{2k+1} = \frac{4}{9}((\frac{9}{4} + \frac{9}{4}i)n - (4+5i)2^{2n} + (12+6i)2^{2n}n + (\frac{25}{4} + 5i))$.

**(c)** $\sum_{k=0}^{n} GG_{2k} = \frac{4}{9}((\frac{9}{4} + \frac{9}{4}i)n - (5+4i)2^{2n} + (6+3i)2^{2n}n + (5+4i))$.

Second, taking $GW_n = GH_n$ with $GH_0 = 3 + 2i, GH_1 = 5 + 3i, GH_2 = 9 + 5i$. We have the following corollary:

**Corollary 5.3.** *(Sum of the Gaussian modified Cullen numbers). For* $n \geq 0$ *we have the following formulas:*

**(a)** $\sum_{k=0}^{n} GH_k = 2^{n+2} + n - 1 + i(n + 2^{n+1})$.
**(b)** $\sum_{k=0}^{n} GH_{2k+1} = \frac{1}{3}(2^{2n+4} + 3n - 1) + i(\frac{1}{3}(2^{2n+3} + 3n + 1))$.
**(c)** $\sum_{k=0}^{n} GH_{2k} = \frac{1}{3}(2^{2n+3} + 3n + 1) + i(n + \frac{1}{3}2^{2n+2} + \frac{2}{3})$.

Third, taking $GW_n = GR_n$ with $GR_0 = -1 - \frac{3}{2}i, GR_1 = 1 - i, GR_2 = 7 + i$. We get the following corollary:

**Corollary 5.4.** *(Sum of the Gaussian Woodall numbers). For $n \geq 0$ we have the following formulas:*

**(a)** $\sum_{k=0}^{n} GR_k = (n-1)(2^{n+1} - 1) + i(2^{n+1}(n-1) - n - 2^n n + \frac{1}{2})$.
**(b)** $\sum_{k=0}^{n} GR_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3} - 9n + 1) + i(\frac{1}{9}((3n-1)2^{2n+3} - 9n - 1))$.
**(c)** $\sum_{k=0}^{n} GR_{2k} = \frac{1}{9}((3n-1)2^{2n+3} - 9n - 1) + i(\frac{1}{9}2^{2n+3}(2n-1) - \frac{1}{9}2^{2n+1}(2n+1) - n - \frac{7}{18})$.

Fourth, taking $GW_n = GC_n$ with $GC_0 = 1 + \frac{1}{2}i, GC_1 = 3 + i, GC_2 = 9 + 3i$. We have the following corollary:

**Corollary 5.5.** *(Sum of the Gaussian Cullen numbers). For $n \geq 0$ we have the following formulas:*

**(a)** $\sum_{k=0}^{n} GC_k = (n-1)2^{n+1} + n + 3 + i(n + 2^{n+1}(n-1) - 2^n n + \frac{5}{2})$.
**(b)** $\sum_{k=0}^{n} GC_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3} + 9n + 19) + i(\frac{1}{9}((3n-1)2^{2n+3} + 9n + 17))$.
**(c)** $\sum_{k=0}^{n} GC_{2k} = \frac{2}{9}((\frac{9}{2} + \frac{9}{2}i)n - (4+5i)2^{2n} + (12+6i)2^{2n}n + (\frac{17}{2} + \frac{29}{4}i))$.

# 6 MATRIX FORMULATION OF $GW_n$

Consider the sequence $\{G_n\}$ which is defined by the third-order recurrence relation

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}$$

with the initial conditions

$$G_0 = 0, G_1 = 1, G_2 = 5.$$

We define the square matrix $A$ of order $3$ as

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. We give the following Lemma.

**Lemma 6.1.** *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

Proof. The Lemma (6.1) equality can be proved by strong induction on $n$. If $n = 0$ we obtain

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n \leq k$. So that the following identity is true.

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{aligned}
\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} \\
&= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} \\
&= \begin{pmatrix} 5GW_{k+2} - 8GW_{k+1} + 4GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.
\end{aligned}$$

Consequently, by induction on $n$, the proof is finished. □

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & G_{n-2} \end{pmatrix}.$$

For the proof see [47].

**Theorem 6.2.** *We assume that the matrices $N_{GW}$ and $E_{GW}$ are defined as follows*

$$N_{GW} = \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix},$$

$$E_{GW} = \begin{pmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{pmatrix}.$$

*The following identity is true between $A^n$, $N_{GW}$ and $E_{GW}$.*

$$A^n N_{GW} = E_{GW}.$$

Proof. Note that one gets

$$\begin{aligned}
A^n N_{GW} &= \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & G_{n-2} \end{pmatrix} \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{aligned}$$

such that

$$\begin{aligned}
a_{11} &= GW_2G_{n+1} + GW_1(4G_{n-1} - 8G_n) + GW_0G_n, \\
a_{12} &= GW_1G_{n+1} + GW_0(4G_{n-1} - 8G_n) + GW_{-1}G_n, \\
a_{13} &= GW_0G_{n+1} + GW_{-1}(4G_{n-1} - 8G_n) + GW_{-2}G_n, \\
a_{21} &= GW_2G_n + GW_1(4G_{n-2} - 8G_{n-1}) + GW_0G_{n-1}, \\
a_{22} &= GW_1G_n + GW_0(4G_{n-2} - 8G_{n-1}) + GW_{-1}G_{n-1}, \\
a_{23} &= GW_0G_n + GW_{-1}(4G_{n-2} - 8G_{n-1}) + GW_{-2}G_{n-1}, \\
a_{31} &= GW_2G_{n-1} + GW_1(4G_{n-3} - 8G_{n-2}) + GW_0G_{n-2}, \\
a_{32} &= GW_1G_{n-1} + GW_0(4G_{n-3} - 8G_{n-2}) + GW_{-1}G_{n-2}, \\
a_{33} &= GW_0G_{n-1} + GW_{-1}(4G_{n-3} - 8G_{n-2}) + GW_{-2}G_{n-2}.
\end{aligned}$$

Using the Theorem (3.6) the proof is completed. □

We have the following identities for $N_{GW}, E_{GW}$ :

$$N_{GG} = \begin{pmatrix} 5+i & 1 & 0 \\ 1 & 0 & \frac{1}{4}i \\ 0 & \frac{1}{4}i & \frac{1}{4}+\frac{1}{2}i \end{pmatrix}, \quad N_{GH} = \begin{pmatrix} 9+5i & 5+3i & 3+2i \\ 5+3i & 3+2i & 2+\frac{3}{2}i \\ 3+2i & 2+\frac{3}{2}i & \frac{3}{2}+\frac{5}{4}i \end{pmatrix},$$
$$N_{GR} = \begin{pmatrix} 7+i & 1-i & -1-\frac{3}{2}i \\ 1-i & -1-\frac{3}{2}i & -\frac{3}{2}-\frac{3}{2}i \\ -1-\frac{3}{2}i & -\frac{3}{2}-\frac{3}{2}i & -\frac{3}{2}-\frac{11}{8}i \end{pmatrix}, \quad N_{GC} = \begin{pmatrix} 9+3i & 3+i & 1+\frac{1}{2}i \\ 3+i & 1+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \\ 1+\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}+\frac{5}{8}i \end{pmatrix}.$$

and

$$E_{GG} = \begin{pmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{pmatrix}, \quad E_{GH} = \begin{pmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{pmatrix},$$
$$E_{GR} = \begin{pmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{pmatrix}, \quad E_{GC} = \begin{pmatrix} GC_{n+2} & GC_{n+1} & GC_n \\ GC_{n+1} & GC_n & GC_{n-1} \\ GC_n & GC_{n-1} & GC_{n-2} \end{pmatrix}.$$

From the previous theorem presents, we have the following corollary.

**Corollary 6.3.** *The following identities are true:*

**(a)** $A^nN_{GG} = E_{GG}$.

**(b)** $A^nN_{GH} = E_{GH}$.

**(c)** $A^nN_{GR} = E_{GR}$.

**(d)** $A^nN_{GC} = E_{GC}$.

# 7 CONCLUSIONS

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers and generalized third-order Pell numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art.

- In section 1, we present some background about the Gaussian generalized Woodall numbers and give some information about Gaussian sequences from literature.
- In section 2, we define Gaussian generalized Woodall numbers and give some proporties such as Binet's formula and generating function.
- In section 3,we present some identities, using

reccurance relation and generating function, on Gaussian modified Woodall, Gaussian modified Cullen, Gaussian Woodall, Gaussian Cullen numbers.

- In section 4, we give Simpson's formula of Gaussian generalized Woodall numbers.
- In section 5, we identify some sum formulas of Gaussian generalized Woodall numbers.
- In section 6, We give the square matrix $A^n$ using modified Woodall sequence $\{G_n\}$ and present some identies abouth Gaussian generalized Woodall numbers.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [3].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [6].
- For the application of Jacobsthal numbers to special matrices, see [71].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [22].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [72].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [73].
- For the applications of generalized Fibonacci numbers to binomial sums, see [70].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [49].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [50].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [14].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [37].
- For the application of higher order Jacobsthal numbers to quaternions, see [38].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [20].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [5].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [29].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [30].
- For the applications of $k$-Fibonacci and $k-$Lucas numbers to spinors, see [32].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [40].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [19].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [13].

We now present some applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [11] and [10], respectively.
- For the application of Tribonacci numbers to special matrices, see 74.
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [41] and 4, respectively.
- For the application of Pell-Padovan numbers to groups, see [15].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [18].
- For the application of Gaussian Tribonacci numbers to various graphs, see [65].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [16].
- For the application of Narayan numbers to finite groups see [31].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [51].

- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [52].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see 53.
- For the application of generalized Tribonacci numbers to Sedenions, see [54].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [55].
- For the application of generalized Tribonacci numbers to circulant matrix, see [56].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [69].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [17].

Next, we now list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [57].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see 58.
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [59].
- For the application of generalized Tetranacci numbers to binomial transform, see [60].

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see [42].
- For the application of generalized Pentanacci numbers to quaternions, see [44].
- For the application of generalized Pentanacci numbers to binomial transform, see [61].

  We now present some applications of second order sequences of polynomials.
- For the application of generalized Fibonacci Polynomials to the summation formulas, see [62].
- For some applications of generalized Fibonacci Polynomials, see [63].

  We now present some applications of third order sequences of polynomials.
- For some applications of generalized Tribonacci Polynomials, see [64].

## COMPETING INTERESTS

Authors have declared that no competing interests exist.

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