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Whole Perfect Vectors and Fermat's Last Theorem

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Abstract

A naïve discussion of Fermat's last theorem conundrum is described. The present theorem's proof is grounded on the well-known properties of sums of powers of the sine and cosine functions, the Minkowski norm definition, and some vector-specific structures.

Keywords

Fermat's Last Theorem, Whole Perfect Vectors, Sine and Cosine Functions, Natural and Rational Vectors, Fermat Vectors

1. Introduction

Time has passed since the Wiles 100-page demonstration of Fermat's last theorem [1]. Meanwhile, our laboratory has been working on several computational aspects of this mentioned theorem. These studies were focused on extending Fermat's theorem to larger dimensions and involving powers of higher natural numbers [2] [3] [4] [5].

Still, as far as the present author knows, it seems that no new alternative proofs of the theorem exist, conforming to the handwritten note left by Fermat about a supposedly straightforward proof of the famous initial self-formulated theorem, except for other points of view recently published [6] [7] [8] [9] [10].

The present paper describes a simple proof of Fermat's last theorem.

2. Whole Perfect Vectors

In three-dimensional vector semispaces, see References [11]-[17], constructed on the non-negative real set: $V_3(\mathbb{R}^+)$, one can define a (*whole*) *perfect* vector: $\langle \mathbf{p}| = (a,b,r)$, for more information, see References [15] [16] [17], when the vector elements meet the following property:

$$\langle \mathbf{p}| \in V_3(\mathbb{R}^+) \wedge \{a,b,r\} \subset \mathbb{R}^+ \wedge 0 < a < b < r \text{ .} \quad (1)$$

In such a semispace, the perfect vectors can be collected into a subset $\mathbf{P}$ of the space: $V_3(\mathbb{R}^+)$, that is: $\forall \langle \mathbf{p}| \in \mathbf{P} \subset V_3(\mathbb{R}^+)$.

3. Minkowski *n*-th Order Norms

In the perfect vector subset $\mathbf{P}$, one can define a Minkowski norm of *n*-th order as follows:

$$\forall \langle \mathbf{p}| = (a,b,r) \in \mathbf{P} : M_n(\langle \mathbf{p}|) = a^n + b^n - r^n \text{ .} \quad (2)$$

Thus, in the subset $\mathbf{P}$, one can suppose contained a vector set with a *Banach-Minkowski metric* associated with a *metric* vector: $\langle \mathbf{m}| = (1,1,-1)$.

A recent general study of Minkowski metric spaces discussed such vector space structure; see References [12] [18] [19].

For the sake of coherence, a 3-dimensional space with a defined Minkowski norm can be named as (2 + 1)-dimensional, for example: $V_{(2+1)}(\mathbb{N})$, notes a natural semispace where one has defined a Minkowski metric, like in Equation (2).

First-Order Minkowski Norms and Fermat's Last Theorem

Before proceeding to the interesting body of the Fermat theorem, it might be worthwhile to discuss the trivial case of Fermat's theorem within the unit power of natural (2 + 1)-dimensional WP vectors. One can write within this restricted first-order situation that:

$$\forall \langle \mathbf{p}| = (a,b,r) \in \mathbf{P} : M_1(\langle \mathbf{p}|) = a + b - r \text{ .} \quad (3)$$

This result means that if one seeks a zero norm, necessarily one has to write:

$$\forall \langle \mathbf{f}_1| = (a,b,a+b) \in \mathbf{P} : M_1(\langle \mathbf{f}_1|) = 0 \text{ ,} \quad (4)$$

therefore, one could locate the first-order Fermat vectors into the infinite symmetric square matrix defined as:

$$\mathbf{S} = \{ s_{IJ} = I + J | \forall I, J \in \mathbb{N} \} \Rightarrow \langle \mathbf{f}_1| = (I, J, s_{IJ}) \leftarrow \forall I, J \in \mathbb{N} \wedge I < J \text{ ,} \quad (5)$$

that contains infinite first-order Fermat natural vectors, but evidences that not all natural WP vectors are first-order Fermat vectors, because there might be some natural vector with $r \neq s_{IJ}$.

Also, for the first-order Fermat vectors, one can write the rational formalism that can be deduced from the structure of the first-order vectors and the Minkowski norm, as shown in Equation (4):

$$\forall \langle \mathbf{f}_1| = (a,b,a+b) \in \mathbf{P} : M_1(\langle \mathbf{f}_1|) = 0 \Rightarrow \frac{a}{a+b} + \frac{b}{a+b} = 1 \text{,} \quad (6)$$

Calling the two terms in the sum (6): $\{p_a, p_b\} \subset [0,1]$, and realizing that they belong to the unit interval is the same as considering the first-order Fermat vectors isomorphic to the set of two-dimensional probability distributions.

4. Homothecy and Original Vector

The homotheties of an *original* perfect vector $\langle \mathbf{p}|$ are defined as:

$$\forall \langle \mathbf{p}| \in \mathbf{P} \wedge \forall \lambda \in \mathbb{R}^{+} : \langle \mathbf{h}| = \lambda \langle \mathbf{p}| = (\lambda a, \lambda b, \lambda r) \in \mathbf{P} . \quad (7)$$

The Minkowski norms of the homotheties $\langle \mathbf{h}|$ of perfect vectors are easily related to the ones associated with a perfect origin vector:

$$M_n(\langle \mathbf{h}|) = (\lambda a)^n + (\lambda b)^n - (\lambda r)^n = \lambda^n (a^n + b^n - r^n) = \lambda^n M_n(\langle \mathbf{p}|). \quad (8)$$

This above equation corresponds to the fact that a vector with a Minkowski norm-specific value has the homothetic vector norms as the original vector one multiplied by a factor equivalent to the homothecy parameter, powered to the order of the norm.

5. Extended Fermat Vectors

A WP vector can be named as an *extended* Fermat vector when its second-order Minkowski norm is null, that is:

$$\forall \langle \mathbf{f}| \in \mathbf{F} \subset \mathbf{P} : M_2(\langle \mathbf{f}|) = 0 \Rightarrow a^2 + b^2 - r^2 = 0 \leftrightarrow a^2 + b^2 = r^2 . \quad (9)$$

6. Natural Fermat Vectors

As natural numbers are a subset of non-negative real numbers: $\mathbb{N} \subset \mathbb{R}^{+}$, there can exist within the extended Fermat vectors subset, some *natural* Fermat vectors with elements belonging entirely to the natural number set. If this is the case, they can be called shortly *true natural* Fermat vectors (of second order or order 2) and symbolize their subset with $\mathbf{T}$, that is:

$$\forall \langle \mathbf{t}| = (a,b,r) \in \mathbf{T} \subset \mathbf{F} \wedge \{a,b,r\} \subset \mathbb{N} : M_2(\langle \mathbf{t}|) = 0 . \quad (10)$$

The so-called Pythagorean triples are a nickname for true Fermat vectors (of second order).

True Fermat Vectors Construction and First Order Minkowski Norms

One can use a similar argument to the one employed in the subsection of Section 3 to obtain a construct of all the possible second order true natural Fermat vectors, but using the first order Minkowski norm.

Let's construct first the set of squares of all the natural numbers: $\mathbb{N}^{[2]} = \{1, 2^2, 3^2, \cdots, I^2, \cdots\} \subset \mathbb{N}$.

Then, one can use the symbols: $\forall z_I \in \mathbb{N}^{[2]} \rightarrow z_I = I^2$, for the elements of the natural squared numbers. Furthermore, one can construct the infinite-dimensional matrix:

$$\mathbf{Z} = \{ z_{IJ} = z_I + z_J = I^2 + J^2 \,|\, I, J \in \mathbb{N} \wedge I < J \}, \quad (11)$$

in this manner, one can see one can construct the true natural Fermat vectors, using the upper triangle of the matrix $\mathbf{Z}$.

To prove this claim, first one can form the vectors $\langle \mathbf{f} | = (z_I, z_J, z_{IJ})$; second, one can pick up only the elements of the matrix $\mathbf{Z}$ belonging to the set of squared natural numbers, then one can write:

$$\begin{aligned} &\forall I < J \wedge z_{IJ} \in \mathbb{N}^{[2]} \wedge \forall \langle \mathbf{f} | = (z_I, z_J, z_{IJ}) \in V_{(2+1)}(\mathbb{N}) \\ &\rightarrow M_1(\langle \mathbf{f} |) = z_I + z_J - z_{IJ} = 0 \end{aligned} \quad (12)$$

which can be seen as an algorithm to construct the true Fermat vectors of second order.

7. Rational Fermat Vectors

On the other hand, any true Fermat vector can be transformed into a vector with elements defined within the non-negative rational number set: $\mathbb{Q}^+ \subset \mathbb{R}^+$. Such a possibility is easy to consider, as it can be written:

$$\begin{aligned} &\forall \langle \mathbf{t} | = (a, b, r) \in \mathbf{T} : M_2(\langle \mathbf{t} |) = 0 \\ &\Rightarrow a^2 + b^2 = r^2 \Rightarrow \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1 \Rightarrow \left\{\left(\frac{a}{r}\right), \left(\frac{b}{r}\right)\right\} \subset \mathbb{Q}^+. \end{aligned} \quad (13)$$

Therefore, the vectors defined over the non-negative rational set have the form:

$$\langle \mathbf{k} | = \left(\left(\frac{a}{r}\right), \left(\frac{b}{r}\right), 1\right) \in \mathbf{K} \rightarrow M_2(\langle \mathbf{k} |) = 0, \quad (14)$$

and could be considered as *extended rational* Fermat vectors, with elements constructed over the set $\mathbb{Q}^+$, whenever Equations (13) and (14) hold.

Therefore, by construction, one can write:

$$0 < a < b < r \leftrightarrow 0 < \frac{a}{r} < \frac{b}{r} < 1, \quad (15)$$

and thus, one can also consider the vectors of the subset $\mathbf{K}$ as possessing elements defined within the $(0,1]$ unit interval.

8. Isomorphism between Natural and Rational Fermat Vectors

In fact, true natural Fermat vectors and rational Fermat vector sets are isomorphic via a homothecy, which can also be accepted as acting like an operator, that is:

$$\forall \langle \mathbf{t} | \in \mathbf{T} : \exists r^{-1} \langle \mathbf{t} | = \langle \mathbf{k} | \in \mathbf{K} \Leftrightarrow \forall \langle \mathbf{k} | \in \mathbf{K} : \exists r \langle \mathbf{k} | = \langle \mathbf{t} | \in \mathbf{T}. \quad (16)$$

One can also symbolically write:

$$r^{-1}(\mathbf{T}) = \mathbf{K} \Leftrightarrow r(\mathbf{K}) = \mathbf{T}. \quad (17)$$

Therefore, proving Fermat's last theorem in the set $\mathbf{T}$ is the same as proving it in the set $\mathbf{K}$, and vice versa.

9. Trigonometric Fermat Vectors

Such an isomorphism between true natural and rational Fermat vectors is essential because the vectors in $\mathbf{K}$ can be rewritten with trigonometric functions.

First, note that as the true Fermat vectors are whole perfect vectors, one can suppose that the relations of Equation (15) hold. Second, because one initially deals with natural Fermat vectors, one can also write:

$$a^2 + b^2 - r^2 = 0 \leftrightarrow \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 - 1 = 0, \tag{18}$$

Then, taking angles in the interval $\alpha \in \left(0, \frac{\pi}{4}\right]$, due to the symmetrical nature of the sine and cosine functions, one can write the true natural Fermat vectors as vectors possessing trigonometric functions as elements, instead of divisions of two natural numbers, that is:

$$\begin{gathered} \forall \alpha \in \left(0, \frac{\pi}{4}\right] \wedge C = \cos(\alpha); S = \sin(\alpha): \\ \langle \mathbf{u}| = (S, C, 1) \in \mathbf{U} \to M_2(\langle \mathbf{u}|) = S^2 + C^2 - 1 = 0 \end{gathered} \tag{19}$$

Some Remarks on Trigonometric Fermat Vectors

Not all the trigonometric vectors $\langle \mathbf{u}| \in \mathbf{U}$ written as in Equation (19) could be associated to the rational Fermat vectors $\langle \mathbf{k}|$.

Trigonometric vectors of type $\langle \mathbf{u}|$ can be seen as extended Fermat vectors. But because all true natural Fermat vectors $\langle \mathbf{t}|$ can generate rational Fermat vectors $\langle \mathbf{k}|$, one can undoubtedly write that:

$$\mathbf{K} \subset \mathbf{U} \Rightarrow r^{-1}(\mathbf{T}) \in \mathbf{U} \to r(\mathbf{K}) = \mathbf{T}. \tag{20}$$

As we have seen, one can transform all true natural Fermat vectors into rational Fermat vectors, which can also be expressed as trigonometric Fermat vectors.

10. Minkowski Norms of Trigonometric Vectors

Then, due to the nature of the expressions of the powers of the sine and cosine functions, and in compliance with Fermat's theorem, one can write that:

$$\begin{gathered} \forall \langle \mathbf{u}| = (S, C, 1) \in \mathbf{U} \\ \to \forall n \in \mathbb{N} \wedge n \neq 2 : M_n(\langle \mathbf{u}|) = S^n + C^n - 1 \neq 0 \end{gathered} \tag{21}$$

Such inequality can be easily proven upon knowing the sum of natural powers expressions of both sine and cosine functions. The most revealing source can be found in Reference [20]. Thus, the formulation will not be explicitly repeated here, except for the 1st, 3rd, and 4th powers, given as a short illustrative example below:

$$\begin{gathered} S + C \neq 1 \\ S^3 + C^3 = \frac{1}{4}\left(3(S + C) + \cos(3\alpha) - \sin(3\alpha)\right) \neq 1 \\ S^4 + C^4 = \frac{1}{4}\left(3 + \cos(4\alpha)\right) \neq 1 \end{gathered} \tag{22}$$

Inequality expressions like those shown in Equation (22) can be easily seen as different from unity *in general*. Sums of larger powers are readily available, yielding terms for the sums of powers, which also clearly differ from the unity.

Therefore, the natural true natural Fermat vectors are isomorphic to *some* trigonometric vectors $\langle\mathbf{u}|$, and their Minkowski norms satisfy the above Equation (21) inequality.

Hence, a Fermat last theorem holds for *all* rational Fermat vectors and thus has to hold for the true natural Fermat vectors because of the isomorphism early discussed between the sets $\mathbf{T}$ and $\mathbf{K}$.

11. Discarding the Existence of True Natural Fermat Vectors of Order Greater than 2

In previous sections, one has implicitly shown Fermat's last theorem. One can suppose such a demonstration included within the definition of true natural Fermat (2 + 1)-dimensional vectors by an attached Minkowski norm. This fact makes the existence of true natural Fermat vectors of order higher than two impossible.

A discussion follows of whether one might construct natural (2 + 1)-dimensional vectors as true natural Fermat vectors of orders higher than the second, as describing a complete Fermat's theorem proof.

Suppose one wants to demonstrate that vectors in any (2 + 1)-dimensional natural vector space with a well-defined Minkowski norm cannot be true natural Fermat vectors of order higher than 2.

That is:

$$\forall n \in \mathbb{N} \wedge n > 2 : \forall \langle\mathbf{p}| = (a,b,r) \in V_{(2+1)}(\mathbb{N}) \Rightarrow M_n(\langle\mathbf{p}|) \neq 0, \tag{23}$$

then one can continue, trying to follow a *reductio ad absurdum* procedure leading to the demonstration.

One can start admitting that Equation (23) is false, so one can write the following property for some natural vector and Minkowski norm order:

$$\exists p \in \mathbb{N} \wedge p > 2 : \exists \langle\mathbf{p}| = (a,b,r) \in V_{(2+1)}(\mathbb{N}) \Rightarrow M_p(\langle\mathbf{p}|) = 0, \tag{24}$$

therefore, if the expression (24) is true, then one can also write:

$$M_p(\langle\mathbf{p}|) = 0 \rightarrow a^p + b^p - r^p = 0 \rightarrow a^p + b^p = r^p \Rightarrow \left(\frac{a}{r}\right)^p + \left(\frac{b}{r}\right)^p = 1 \tag{25}$$

in the same manner, one can use the above equalities also to write:

$$x = \frac{a}{r} \wedge y = \frac{b}{r} \rightarrow x^p + y^p = 1. \tag{26}$$

However, the pair of rational numbers: $\{x, y\} \subset \mathbb{Q}^+$, corresponds to the Cartesian coordinates of a point situated into a circumference of unit radius.

One can admit such a previous affirmation following a similar reasoning as in Section 9. Choosing the appropriate angle in trigonometric coordinates, one can write:

$$\{x, y\} \to \{\sin(\alpha), \cos(\alpha)\} \equiv \{S, C\}. \quad (27)$$

Then, owing to Equation (27), one can also write:

$$S^2 + C^2 = 1 \to x^2 + y^2 = 1 \Rightarrow \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1 \to M_2\left(\langle\mathbf{p}|\right) = 0, \quad (28)$$

therefore, the vector $\langle\mathbf{p}|$ of Equation (23) is a rational Fermat vector. Such a result contradicts the existence of Minkowski norms higher than 2, as expressed in Equation (24).

Therefore, one cannot obtain natural vectors fulfilling Equation (24) providing Minkowski norms of order larger than two; thus, Equation (23) must be true.

As a result, one can say that true natural Fermat vectors of order higher than two cannot exist.

Moreover, this implies that in the context of the present study, only true natural Fermat vectors of dimension (2 + 1) and order two are relevant.

12. Discussion

One can write that some whole perfect natural vectors fulfill the equation concerning the nullity of the second-order Minkowski norm:

$$\exists\langle\mathbf{t}| = (a, b, r) \in \mathbf{T} : M_2\left(\langle\mathbf{t}|\right) = 0 \to \langle\mathbf{t}| = \langle\mathbf{f}|, \quad (29)$$

defining in this way true Fermat vectors of second order.

Here, in general, one has deduced, via the isomorphism between natural and rational Fermat vectors, under a trigonometric representation, that true natural Fermat vectors cannot possess Minkowski null norms other than the second-order ones:

$$\forall n \in \mathbb{N} \wedge n > 2 \wedge \forall\langle\mathbf{v}| = (a, b, r) \in V_{(2+1)}(\mathbb{N}) \Rightarrow M_n\left(\langle\mathbf{v}|\right) \neq 0. \quad (30)$$

A result corresponding to how one can reformulate Fermat's Last Theorem.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Wiles, A. (1995) Modular Elliptic-Curves and Fermat's Last Theorem. *Annals of Mathematics*, **141**, 443-551. https://doi.org/10.2307/2118559

[2] Carbó-Dorca, R. (2017) Natural Vector Spaces (Inward Power and Minkowski Norm of a Natural Vector, Natural Boolean Hypercubes) and Fermat's Last Theorem. *Journal of Mathematical Chemistry*, **55**, 914-940. https://doi.org/10.1007/s10910-016-0708-6

[3] Carbó-Dorca, R., Muñoz-Caro, C., Niño, A. and Reyes, S. (2017) Refinement of a Generalized Fermat's Last Theorem Conjecture in Natural Vector Spaces. *Journal of Mathematical Chemistry*, **55**, 1869-1877. https://doi.org/10.1007/s10910-017-0766-4

[4] Niño, A., Reyes, S. and Carbó-Dorca, R. (2021) An HPC Hybrid Parallel Approach to the Experimental Analysis of Fermat's Theorem Extension to Arbitrary Dimensions on Heterogeneous Computer Systems. *The Journal of Supercomputing*, **77**, 11328-11352. https://doi.org/10.1007/s11227-021-03727-2

[5] Carbó-Dorca, R., Reyes, S. and Niño, A. (2021) Extension of Fermat's Last Theorem in Minkowski Natural Spaces. *Journal of Mathematical Chemistry*, **59**, 1851-1863. https://doi.org/10.1007/s10910-021-01267-x

[6] Ossicini, A. (2022) On the Nature of Some Euler's Double Equations Equivalent to Fermat's Last Theorem. *Mathematics*, **10**, 4471-4483. https://doi.org/10.3390/math10234471

[7] Klykov, S.P. (2023) Elementary Proofs for the Fermat Last Theorem in Z Using One Trick for a Restriction in Z_p. *Journal of Science and Arts*, **23**, 603-608. https://doi.org/10.46939/J.Sci.Arts-23.3-a03

[8] Klykov, S.P. and Klykova, M. (2023) An Elementary Proof of Fermat's Last Theorem. https://doi.org/10.13140/RG.2.2.19455.59044

[9] Gilbert, J.B. (2023) A Proof of Fermat's Last Theorem. https://doi.org/10.13140/RG.2.2.27051.82722

[10] Castro, C. (2023) Finding Rational Points of Circles, Spheres, Hyper-Spheres via Stereographic Projection and Quantum Mechanics. https://doi.org/10.13140/RG.2.2.12030.36164

[11] Carbó-Dorca, R. (1998) Fuzzy Sets and Boolean Tagged Sets, Vector Semispaces and Convex Sets, QSM and ASA Density Functions, Diagonal Vector Spaces and Quantum Chemistry. In: Carbó-Dorca, R. and Mezey, P.G., Eds., *Advances in Molecular Similarity*, Vol. 2, JAI Press, London, 43-72. https://doi.org/10.1016/S1873-9776(98)80008-4

[12] Carbó-Dorca, R. (2002) Shell Partition and Metric Semispaces: Minkowski Norms, Root Scalar Products, Distances and Cosines of Arbitrary Order. *Journal of Mathematical Chemistry*, **32**, 201-223. https://doi.org/10.1023/A:1021250527289

[13] Bultinck, P. and Carbó-Dorca, R. (2004) A Mathematical Discussion on Density and Shape Functions, Vector Semispaces and Related Questions. *Journal of Mathematical Chemistry*, **36**, 191-200. https://doi.org/10.1023/B:JOMC.0000038793.21806.65

[14] Carbó-Dorca, R. (2008) Molecular Quantum Similarity Measures in Minkowski Metric Vector Semispaces. *Journal of Mathematical Chemistry*, **44**, 628-636. https://doi.org/10.1007/s10910-008-9442-z

[15] Carbó-Dorca, R. (2019) Role of the Structure of Boolean Hypercubes When Used as Vectors in Natural (Boolean) Vector Semispaces. *Journal of Mathematical Chemistry*, **57**, 697-700. https://doi.org/10.1007/s10910-018-00997-9

[16] Carbó-Dorca, R. (2018) Boolean Hypercubes and the Structure of Vector Spaces. *Journal of Mathematical Sciences and Modelling*, **1**, 1-14. https://doi.org/10.33187/jmsm.413116

[17] Carbó-Dorca, R. (2022) Shadows' Hypercube, Vector Spaces, and Non-Linear Optimi-

zation of QSPR Procedures. *Journal of Mathematical Chemistry*, **60**, 283-310. https://doi.org/10.1007/s10910-021-01301-y

[18] Carbó-Dorca, R. and Chakraborty, T. (2021) Extended Minkowski Spaces, Zero Norms, and Minkowski Hypersurfaces. *Journal of Mathematical Chemistry*, **59**, 1875-1879. https://doi.org/10.1007/s10910-021-01266-y

[19] Carbó-Dorca, R. (2021) Generalized Scalar Products in Minkowski Metric Spaces. *Journal of Mathematical Chemistry*, **59**, 1029-1045. https://doi.org/10.1007/s10910-021-01229-3

[20] Weisstein, E.W. Trigonometric Power Formulas. From MathWorld: A Wolfram Web Resource. https://mathworld.wolfram.com/TrigonometricPowerFormulas.html