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*Corrigendum*

# Corrigendum to "Smarandache Ruled Surfaces according to Frenet-Serret Frame of a Regular Curve in $E^3$"

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In this paper, we introduce original definitions of Smarandache ruled surfaces according to Frenet-Serret frame of a curve in $E^3$. It concerns TN-Smarandache ruled surface, TB-Smarandache ruled surface, and NB-Smarandache ruled surface. We investigate theorems that give necessary and sufficient conditions for those special ruled surfaces to be developable and minimal. Furthermore, we present examples with illustrations.

In the article titled "Smarandache Ruled Surfaces according to Frenet-Serret Frame of a Regular Curve in $E^3$" [1], the numbering of equations is incorrect. These have been corrected in the revised version shown below:

## 1. Introduction

In differential geometry of curves and surfaces [1–3], a ruled surface defines the set of a family of straight lines depending on a parameter. The straight lines mentioned are the rulings of the ruled surface. The general parametric representation of a ruled surface is $\Psi(s, v) = c(s) + v\overleftarrow{X}(s)$, where $c(s)$ is a curve through which all rulings pass; it is called the base curve of the surface; the vector $\overleftarrow{X}(s)$ defines the ruling direction.

It is well known that ruled surfaces are of great interest to many applications and have contributed in several areas, such as mathematical physics, kinematics and Computer Aided Geometric Design (CAGD).

Ruled surfaces have been studied differently by an important number of researchers. In [4], the authors constructed the ruled surface whose rulings are constant linear combinations of alternative moving frame vectors of its base curve; they studied the ruled surface properties, characterize it, and presented examples with illustrations in the case of general helices [5] and slant helices [6] as base curves. In [7], the authors constructed the ruled surface whose rulings are constant linear combinations of Darboux frame vectors of a regular curve lying on a regular surface of reference in $E^3$. Their point of interest was to make a comparative study between the two surfaces (the regular surface of reference and the new constructed ruled surface) along their common curve. Also, they investigated properties of the constructed ruled surface. Moreover, they gave examples with illustrations.

The notions of developability and minimalist are two of the most important properties of surfaces.

The ruled surfaces with vanishing Gaussian curvature, which can be transformed into the plane without any deformation and distortion, are called developable surfaces; they form a relatively small subset that contains cylinders, cones, and the tangent surfaces [8–10].

A minimal surface is a surface that locally minimizes its area. It is referred to the fixed boundary curve of a surface area that is minimal with respect to other surfaces with the same boundary. This is equivalent to having zero mean curvature [11–13].

In curve theory, *Smarandache curves* are one of the special innovated curves that were introduced at the first time in Minkowski space time by the authors in [14]. It is about curves whose position vectors are composed by Frenet-Serret frame vectors on another regular curve. In [15, 16, 18], we can find several research works about *Smarandache* *curves* according to different frames such as Frenet-Serret frame, Bishop frame, and Darboux frame in Euclidean and Minkowski spaces.

The motivation of this work is inspired by ruled surface and Smarandache curve. We are eager to introduce new definitions that combine those two important notions and study their properties. We are also opening up opportunities to perceive future works that are relative to applications in differential geometry, physics, and medical science.

In this paper, we are interested in ruled surfaces generated by Smarandache curves according to Frenet-Serret frame. Indeed, we construct and introduce original definitions of three special ruled surfaces generated by TN-Smarandache curve, TB-Smarandache curve, and NB-Smarandache curve according to Frenet-Serret frame of an arbitrary regular curve in $E^3$. We investigate theorems that give us necessary and sufficient conditions for those three ruled surfaces to be developable and minimal. Finally, we give examples with illustrations.

## 2. Preliminaries

In the Euclidean 3-space $E^3$, we consider the usual metric given by

$$\langle , \rangle = dx_1 + dx_2 + dx_3, \tag{1}$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $E^3$.

A curve $c : s \in I \subset \mathbb{R} \longrightarrow E^3$ is said to be of unit speed (or parameterized by the arc-length) if $\|c'(s)\| = \sqrt{\langle c'(s), c'(s)\rangle} = 1$ for any $s \in I$. For such curves, there is a frame $\{\overleftarrow{T}, \overleftarrow{N}, \overleftarrow{B}\}$ which is called the Frenet-Serret frame, where $\overleftarrow{T} = c'$,$\overleftarrow{N} = c''/\|c''\|$, and $\overleftarrow{B} = \overleftarrow{T} \times \overleftarrow{N}$ are the unit tangent, the principal normal, and the binormal vector of the curve, respectively. The Frenet-Serret formulae of $c(s)$ is given by

$$\begin{bmatrix} \overleftarrow{T}' \\ \overleftarrow{N}' \\ \overleftarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \overleftarrow{T} \\ \overleftarrow{N} \\ \overleftarrow{B} \end{bmatrix}, \tag{2}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of $c(s)$ and they are given by $\kappa = \|c''\|$ and $\tau = \det(c', c'', c''')/\|c''\|^2$, respectively [2].

*Definition 1* (see [17]). TN-Smarandache curves according to Frenet-Serret frame of the curve $c = c(s)$ are given by

$$\beta(s^*(s)) = \frac{1}{\sqrt{2}}\left(\overleftarrow{T}(s) + \overleftarrow{N}(s)\right). \tag{3}$$

*Definition 2* (see [17]). TB-Smarandache curves according to Frenet-Serret frame of the curve $c = c(s)$ are given by

$$\gamma(s^*(s)) = \frac{1}{\sqrt{2}}\left(\overleftarrow{T}(s) + \overleftarrow{B}(s)\right). \tag{4}$$

*Definition 3* (see [17]). NB-Smarandache curves according to Frenet-Serret frame of the curve $c = c(s)$ are given by

$$\delta(s^*(s)) = \frac{1}{\sqrt{2}}\left(\overleftarrow{N}(s) + \overleftarrow{B}(s)\right). \tag{5}$$

Let $\Psi : (s, v) \mapsto c(s) + v\overleftarrow{X}(s)$ be a ruled surface in $E^3$.

Let it denoted by $\overleftarrow{n} = \overleftarrow{n}(s, v)$, the unit normal on ruled surface $\Psi$ at a regular point $\Psi(s, v)$, we have

$$\overleftarrow{n} = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|} = \frac{\left(c' + v\overleftarrow{X}'\right) \times \overleftarrow{X}}{\left\|\left(c' + v\overleftarrow{X}'\right) \times \overleftarrow{X}\right\|}, \tag{6}$$

where $\Psi_s = \partial\Psi(s, v)/\partial s$ and $\Psi_v = \partial\Psi(s, v)/\partial v$.

*Definition 4* (see [19]). The distribution parameter $\lambda = \lambda(s)$ of ruled surface $\Psi$ is given by

$$\lambda = \frac{\det\left(c', \overleftarrow{X}, \overleftarrow{X}'\right)}{\left\|\overleftarrow{X}'\right\|^2}. \tag{7}$$

*Definition 5* (see [19]). A ruled surface is developable if its distribution parameter vanishes.

The first I and the second II fundamental forms of ruled surface $\Psi$ at a regular point $\Psi(s, v)$ are defined, respectively, by

$$\begin{aligned} I(\Psi_s ds + \Psi_v dv) &= E ds^2 + 2F ds dv + G dv^2, \\ II(\Psi_s ds + \Psi_v dv) &= e ds^2 + 2f ds dv + g dv^2, \end{aligned} \tag{8}$$

where

$$E = \|\Psi_s\|^2, F = \langle\Psi_s, \Psi_v\rangle, G = \|\Psi_v\|^2, e = \left\langle\Psi_{ss}, \overleftarrow{n}\right\rangle, f = \left\langle\Psi_{sv}, \overleftarrow{n}\right\rangle, g = \left\langle\Psi_{vv}, \overleftarrow{n}\right\rangle = 0. \tag{9}$$

*Definition 6.* The Gaussian curvature $K$ and the mean curvature $H$ of ruled surface $\Psi$ at a regular point $\Psi(s, v)$ are given, respectively, by:

$$K = -\frac{f^2}{EG - F^2}, H = \frac{Ge - 2Ff}{2(EG - F^2)}. \tag{10}$$

**Proposition 7** (see [19]). *A ruled surface is developable if and only if its Gaussian curvature vanishes.*

**Proposition 8** (see [19]). *A regular surface is minimal if and only if its mean curvature vanishes.*

## 3. Smarandache Ruled Surfaces according to Frenet-Serret Frame of a Regular Curve in $E^3$

In the first step, we start our section by giving the following new definitions of Smarandache ruled surfaces according to Frenet-Serret frame of a curve in $E^3$.

*Definition 9.* Let be $c = c(s)$ the $C^2$-class differentiable unit speed curve whose Frenet-Serret apparatus is $\{\overleftarrow{T}(s), \overleftarrow{N}(s), \overleftarrow{B}(s), \kappa(s), \tau(s)\}$ in $E^3$. The ruled surfaces generated by Smarandache curves according to Frenet-Serret of $c = c(s)$ are as follows:

$$
{}^{1}\Psi(s,v) = \frac{1}{\sqrt{2}}\left(\overleftarrow{T}(s) + \overleftarrow{N}(s)\right) + v\overleftarrow{B}(s), \tag{11}
$$

$$
{}^{2}\Psi(s,v) = \frac{1}{\sqrt{2}}\left(\overleftarrow{T}(s) + \overleftarrow{B}(s)\right) + v\overleftarrow{N}(s), \tag{12}
$$

$$
{}^{3}\Psi(s,v) = \frac{1}{\sqrt{2}}\left(\overleftarrow{N}(s) + \overleftarrow{B}(s)\right) + v\overleftarrow{T}(s). \tag{13}
$$

These ruled surfaces are called TN-Smarandache ruled surface, TB-Smarandache ruled surface, and NB-Smarandache ruled surface, according to Frenet-Serret frame of the curve $c = c(s)$, respectively.

In the following of this section, we investigate theorems that give necessary and sufficient conditions for Smarandache ruled surfaces (11), (12), and (13) to be developable and minimal. Also, we present example with illustration for each Smarandache ruled surface.

**Theorem 10.** *TN-Smarandache ruled surface (11) is developable if and only if $c = c(s)$ is a plane curve.*

**Theorem 11.** *TN-Smarandache ruled surface (11) is minimal if and only if the curvature $\kappa$ and the torsion $\tau$ of the curve $c = c(s)$ satisfy the equation $\kappa(\tau^2 - \kappa' - 2\kappa^2) + v\sqrt{2}(\kappa'\tau + 2\kappa^2\tau) - v^2 2\kappa\tau^2 = 0$.*

*Proof.* Differentiating (11) with respect to $s$ and $v$, respectively, we get

$$
\begin{cases}
{}^{1}\Psi_{s} = -\dfrac{\kappa}{\sqrt{2}}\overleftarrow{T} + \left(\dfrac{\kappa}{\sqrt{2}} - v\tau\right)\overleftarrow{N} + \dfrac{\tau}{\sqrt{2}}\overleftarrow{B}, \\
{}^{1}\Psi_{v} = \overleftarrow{B}.
\end{cases} \tag{14}
$$

The crossproduct of these two vectors gives the normal vector on TN-Smarandache ruled surface (11):

$$
{}^{1}\Psi_{s} \times {}^{1}\Psi_{v} = \left(\frac{\kappa}{\sqrt{2}} - v\tau\right)\overleftarrow{T} + \frac{\kappa}{\sqrt{2}}\overleftarrow{N}, \tag{15}
$$

So, under regularity condition, the unit normal takes the form:

$$
\frac{{}^{1}\Psi_{s} \times {}^{1}\Psi_{v}}{\|{}^{1}\Psi_{s} \times {}^{1}\Psi_{v}\|} = \frac{1}{\sqrt{\kappa^2 - \sqrt{2}\kappa\tau v + \tau^2 v^2}}\left[\left(\frac{\kappa}{\sqrt{2}} - v\tau\right)\overleftarrow{T} + \frac{\kappa}{\sqrt{2}}\overleftarrow{N}\right]. \tag{16}
$$

□

Making the norms for (14), we get the components of the first fundamental form of TN-Smarandache ruled surface (11), at regular points, as follows:

$$
\begin{cases}
{}^{1}E = \dfrac{2\kappa^2 + \tau^2}{2} - \sqrt{2}\kappa\tau v + \tau^2 v^2, \\
{}^{1}F = \dfrac{\tau}{\sqrt{2}}, \\
{}^{1}G = 1.
\end{cases} \tag{17}
$$

Differentiating (14) with respect to $s$ and $v$, respectively, we get

$$
\begin{cases}
{}^{1}\Psi_{ss} = -\left[\dfrac{\kappa'}{\sqrt{2}} + \kappa\left(\dfrac{\kappa}{\sqrt{2}} - v\tau\right)\right]\overleftarrow{T} + \left[\dfrac{-\kappa^2 - \tau^2 + \kappa'}{\sqrt{2}} - v\tau'\right]\overleftarrow{N} + \left[\dfrac{\tau'}{\sqrt{2}} + \tau\left(\dfrac{\kappa}{\sqrt{2}} - v\tau\right)\right]\overleftarrow{B}, \\
{}^{1}\Psi_{vs} = -\tau\overleftarrow{N}, \\
{}^{1}\Psi_{vv} = 0.
\end{cases} \tag{18}
$$

Hence, from (16) to (18), we get the components of the second fundamental form of TN-Smarandache ruled surface (11), at regular points, as follows:

$$
\begin{cases}
{}^{1}e = -\dfrac{\left(\left(\kappa/\sqrt{2}\right) - v\tau\right)\left(\left(\left(\kappa' + \kappa^2\right)/\sqrt{2}\right) - v\kappa\tau\right) + \left(\kappa\left(\kappa^2 + \tau^2\right)/2\right)}{\sqrt{\kappa^2 - \sqrt{2}\kappa\tau v + \tau^2 v^2}}, \\
{}^{1}f = -\dfrac{\kappa\tau}{\sqrt{2}\sqrt{\kappa^2 - \sqrt{2}\kappa\tau v + \tau^2 v^2}}, \\
{}^{1}g = 0.
\end{cases} \tag{19}
$$

From (17) to (19), we get the Gaussian curvature and the mean curvature of TN-Smarandache ruled surface (11), at regular points, as follows:

$$
\begin{cases}
{}^{1}K = -\left[\dfrac{\kappa\tau}{\sqrt{2}\left(\kappa^2 - \sqrt{2}\kappa\tau v + \tau^2 v^2\right)}\right]^2, \\
{}^{1}H = \dfrac{-\kappa\kappa' - 2\kappa^3 + \kappa\tau^2 + v\sqrt{2}\left(\kappa'\tau + 2\kappa^2\tau\right) - v^2 2\kappa\tau^2}{4\left(\kappa^2 - \sqrt{2}\kappa\tau v + \tau^2 v^2\right)^{3/2}},
\end{cases} \tag{20}
$$

which replies to both above theorems.

*Example 1.* Let us consider the regular plane curve ${}^{1}c(s) = (3s - s^3, 3s^2, 0)$. TN-Smarandache ruled surface according to Frenet-Serret frame of the curve ${}^1c(s)$ is given by

$$ {}^1\Psi(s,v)=\frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{1-s^2}{\sqrt{(1-s^2)^2+4s^2}}-\dfrac{s}{\sqrt{1+s^2}} \\ \dfrac{2s}{\sqrt{(1-s^2)^2+4s^2}}+\dfrac{1}{\sqrt{1+s^2}} \\ 0 \end{pmatrix}+v\begin{pmatrix} 0 \\ 0 \\ \dfrac{1-s+2s^2}{\sqrt{\left[(1-s^2)^2+4s^2\right]\left[\sqrt{1+s^2}\right]}} \end{pmatrix}. \quad (21) $$

It is developable but not minimal at its regular points. The following Figure 1 is the illustration of (21) drawn for $(s,v)\in\,]-2\pi,2\pi[\times]-4,4[$.

**Theorem 12.** *TB-Smarandache ruled surface (12) is developable.*

**Theorem 13.** *TB-Smarandache ruled surface (12) is minimal if and only if the curve $c=c(s)$ is a general helix.*

*Proof.* Differentiating (12) with respect to $s$ and $v$, respectively, we get

$$ \begin{cases} {}^2\Psi_s=-v\kappa\overleftarrow{T}+\dfrac{\kappa-\tau}{\sqrt{2}}\overleftarrow{N}+v\tau\overleftarrow{B}, \\ {}^2\Psi_v=\overleftarrow{N}, \end{cases} \quad (22) $$

which implies that the normal vector on TB-Smarandache ruled surface (12) is

$$ {}^2\Psi_s\times{}^2\Psi_v=-v\tau\overleftarrow{T}-v\kappa\overleftarrow{B}, \quad (23) $$

So, under regularity condition, the unit normal is given by

$$ \frac{{}^2\Psi_s\times{}^2\Psi_v}{\|{}^2\Psi_s\times{}^2\Psi_v\|}=\mp\frac{\tau\overleftarrow{T}+\kappa\overleftarrow{B}}{\sqrt{(\kappa^2+\tau^2)}}. \quad (24) $$

Making the norms for (22), we get the components of the first fundamental form of TB-Smarandache ruled surface (12), at regular points

$$ \begin{cases} {}^2E=\dfrac{(\kappa-\tau)^2}{2}+v^2\left(\kappa^2+\tau^2\right), \\ {}^2F=\dfrac{\kappa-\tau}{\sqrt{2}}, \\ {}^2G=1. \end{cases} \quad (25) $$

Differentiating (22) with respect to $s$ and $v$, respectively, we get

$$ \begin{cases} {}^2\Psi_{ss}=-\left[\dfrac{\kappa(\kappa-\tau)}{\sqrt{2}}+v\kappa'\right]\overleftarrow{T}+\left[\dfrac{(\kappa'-\tau')}{\sqrt{2}}-v(\kappa^2+\tau^2)\right]\overleftarrow{N}+\left[\dfrac{\tau(\kappa-\tau)}{\sqrt{2}}+v\tau'\right]\overleftarrow{B}, \\ {}^2\Psi_{vs}=-\kappa\overleftarrow{T}+\tau\overleftarrow{B}, \\ {}^2\Psi_{vv}=0. \end{cases} \quad (26) $$

Hence, from (24) to (26), we get the components of the second fundamental form of TB-Smarandache ruled surface (12) at regular points.

$$ \begin{cases} {}^2e=\mp v\dfrac{\kappa^2(\tau/\kappa)'}{\sqrt{\kappa^2+\tau^2}}, \\ {}^2f=0, \\ {}^2g=0. \end{cases} \quad (27) $$

From (25) to (27), we get the Gaussian curvature and the mean curvature of TB-Smarandache ruled surface (12), at regular points, as follows

$$ \begin{cases} {}^2K=0, \\ {}^2H=\mp\dfrac{\kappa^2(\tau/\kappa)'}{2v(\kappa^2+\tau^2)^{3/2}}, \end{cases} \quad (28) $$

which replies to both above theorems. □

**Corollary 14.** *If $c(s)$ is a general helix, TB-Smarandache ruled surface (12) is developable and minimal.*

*Example 2.* Here, we chose the regular curve ${}^2\mathrm{c(s)}=1/\sqrt{2}(\sin(\sqrt{2}\mathrm{s})/\sqrt{2},-\cos(\sqrt{2}\mathrm{s})/\sqrt{2},\mathrm{s})$ as a general helix. Then, TB-Smarandache ruled surface according to Frenet-Serret frame of ${}^2c(s)$ is:

$$ {}^2\Psi(s,v)=\frac{1}{2}\begin{pmatrix} \cos\left(\sqrt{2}s\right)-\sin\left(\sqrt{2}s\right) \\ \cos\left(\sqrt{2}s\right)+\sin\left(\sqrt{2}s\right) \\ 1 \end{pmatrix}+v\begin{pmatrix} -\cos\left(\sqrt{2}s\right) \\ -\sin\left(\sqrt{2}s\right) \\ 1 \end{pmatrix}. \quad (29) $$

It is developable and minimal at its regular points. The following Figure 2 is the illustration of (29) drawn for $(s,v)\in\,]-2\pi,2\pi[\times]-4,4[$.

**Theorem 15.** *NB-Smarandache ruled surface (13) is developable if and only if $c=c(s)$ is a plane curve.*

**Theorem 16.** *NB-Smarandache ruled surface (13) is minimal if and only if the curvature $\kappa$ and the torsion $\tau$ of the curve $c=c(s)$ satisfy the equation $\tau((2\kappa^2\tau-2\tau^2-\kappa^2)/\sqrt{2})+v(\kappa'\tau+2\kappa\tau^2-\kappa^2\tau')-\sqrt{2}v^2\kappa^2\tau=0$.*

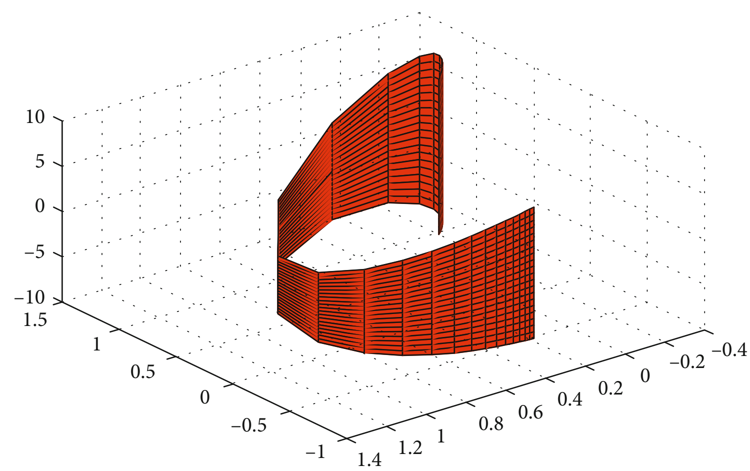


Figure 1: The TN-Smarandache ruled surface (Equation (21)).

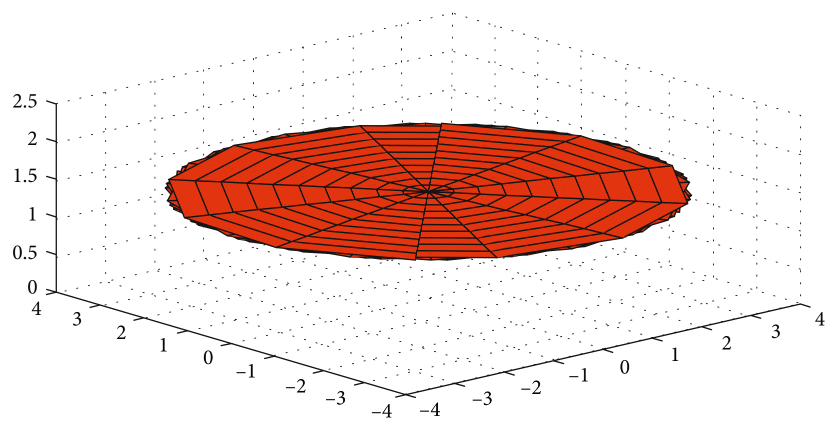


Figure 2: The TB-Smarandache ruled surface (Equation (29)).

*Proof.* Differentiating (13) with respect to $s$ and $v$, respectively, we get

$$\begin{cases} {}^3\Psi_s = -\dfrac{\kappa}{\sqrt{2}}\overleftarrow{T} + \left(-\dfrac{\tau}{\sqrt{2}} + v\kappa\right)\overleftarrow{N} + \dfrac{\tau}{\sqrt{2}}\overleftarrow{B}, \\ {}^3\Psi_v = \overleftarrow{T}, \end{cases} \tag{30}$$

Realizing the crossproduct of those two vectors, we get the normal vector on NB-Smarandache ruled surface (13):

$${}^3\Psi_s \times {}^3\Psi_v = \frac{\tau}{\sqrt{2}}\overleftarrow{N} + \left(\frac{\tau}{\sqrt{2}} - v\kappa\right)\overleftarrow{B}, \tag{31}$$

So, under regularity condition, the unit normal takes the form:

$$\frac{{}^3\Psi_s \times {}^3\Psi_v}{\| {}^3\Psi_s \times {}^3\Psi_v \|} = \frac{1}{\sqrt{\tau^2/2 + \left(\left(\tau/\sqrt{2}\right) - v\kappa\right)^2}}\left[\frac{\tau}{\sqrt{2}}\overleftarrow{N} + \left(\frac{\tau}{\sqrt{2}} - v\kappa\right)\overleftarrow{B}\right]. \tag{32}$$

Applying the norms and the scalar product for both vectors (30), we get the components of the first fundamental form of NB-Smarandache ruled surface (13), at regular points, as follows:

$$\begin{cases} {}^3E = \dfrac{\kappa^2 + \tau^2}{2} + \left(-\dfrac{\tau}{\sqrt{2}} + v\kappa\right)^2, \\ {}^3F = -\dfrac{\kappa}{\sqrt{2}}, \\ {}^3G = 1. \end{cases} \tag{33}$$

Differentiating (30) with respect to $s$ and $v$, respectively, we get

$$\begin{cases} {}^3\Psi_{ss} = -\left[\dfrac{\kappa' - \kappa\tau}{\sqrt{2}} + v\kappa^2\right]\overleftarrow{T} + \left[-\dfrac{\kappa^2 + \tau^2 + \tau'}{\sqrt{2}} + v\kappa'\right]\overleftarrow{N} + \left[\dfrac{\tau' - \tau^2}{\sqrt{2}} + v\kappa\tau\right]\overleftarrow{B}, \\ {}^3\Psi_{vs} = \kappa\overleftarrow{N}, \\ {}^3\Psi_{vv} = 0. \end{cases} \tag{34}$$

Hence, from (32) to (34), we get the components of the second fundamental form of NB-Smarandache ruled surface

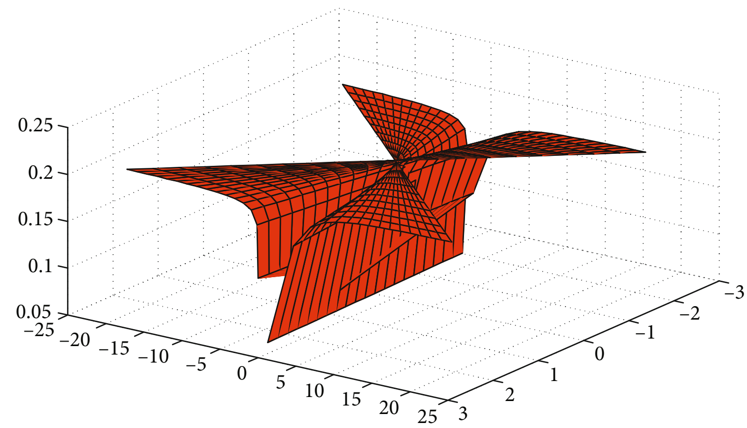


Figure 3: The NB-Smarandache ruled surface (Equation (37)).

(13), at regular points, as follows:

$$\begin{cases} {}^3e = \dfrac{\tau/\sqrt{2}\left[-\left(\kappa^2+\tau^2+\tau'/\sqrt{2}\right)+\nu\kappa'\right]+\left(\tau/\sqrt{2}-\nu\kappa\right)\left[\left(\left(\tau'-\tau^2\right)/\sqrt{2}\right)+\nu\kappa\tau\right]}{\sqrt{\tau^2/2+\left(\tau/\sqrt{2}-\nu\kappa\right)^2}}, \\ {}^3f = \dfrac{\kappa\tau}{\sqrt{\tau^2+\left(\tau-\sqrt{2}\nu\kappa\right)^2}}, \\ {}^3g = 0. \end{cases} \tag{35}$$

From (33) to (35), we get the Gaussian curvature and the mean curvature of NB-Smarandache ruled surface (13), at regular points, as follows:

$$\begin{cases} {}^3K = -\left[\dfrac{\sqrt{2}\kappa\tau}{\tau^2+\left(\tau-\sqrt{2}\nu\kappa\right)^2}\right]^2, \\ \text{1c1c1c} \\ {}^3H = 2\dfrac{\left[\tau\left(\left(2\kappa^2\tau-2\tau^2-\kappa^2\right)/\sqrt{2}\right)\right]+\left[\nu\left(\kappa'\tau+2\kappa\tau^2-\kappa^2\tau'\right)-\sqrt{2}\nu^2\kappa^2\tau\right]}{\left[\tau^2+\left(\tau-\sqrt{2}\nu\kappa\right)^2\right]^{3/2}}, \end{cases} \tag{36}$$

which replies to both above theorems. □

**Corollary 17.** *If NB-Smarandache ruled surface (13) is developable, it is minimal.*

*Example 3.* Let us consider the plane curve ${}^3c(s) = (s^2, s^3, 0)$. Thus, NB-Smarandache ruled surface according to Frenet-Serret frame of ${}^3c(s)$ is parametrized by

$$\Psi(s,\nu) = \frac{1}{\sqrt{2(4+36s^2)}}\begin{pmatrix} 2 \\ 6s \\ \dfrac{6s^2}{\sqrt{4s^2+9s^4}} \end{pmatrix} + \frac{\nu}{\sqrt{4s^2+9s^4}}\begin{pmatrix} 2s \\ 3s^2 \\ 0 \end{pmatrix}. \tag{37}$$

It is developable and minimal. The following Figure 3 is the illustration of (37) which is drawn for $(s,\nu) \in ]-2\pi, 2\pi[ \times ]-4, 4[$.

## 4. Conclusion

*The main results of the present work assure that:*

TN-Smarandache ruled surface according to Frenet-Serret frame is developable (resp., minimal) if $c(s)$ is a plane curve (resp. $\kappa$ and $\tau$ satisfy special equation)

TB-Smarandache ruled surface according to Frenet-Serret frame is developable. However, it is minimal if $c(s)$ is a general helix

NB-Smarandache ruled surface according to Frenet-Serret frame is developable (resp., minimal) if $c(s)$ is a plane curve (resp. $\kappa$ and $\tau$ satisfy special equation)

The investigated theorems prove that developability and minimality conditions of the introduced *Smarandache ruled surfaces* according to Frenet-Serret frame are related to differential properties of the reference curve. Therefore, we can easily constat that to construct a developable Smarandache ruled surface or a minimal Smarandache ruled surface according to Frenet-Serret frame, we just need to make the right choice for the reference curve $c = c(s)$.

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