Hindawi
Abstract and Applied Analysis
Volume 2022, Article ID 6721360, 49 pages
https://doi.org/10.1155/2022/6721360

Research Article

Solutions of a Class of Multiplicatively Advanced Differential Equations II: Fourier Transforms

David W. Pravica, Njinasoa Randriampiry, and Michael J. Spurr

Department of Mathematics, East Carolina University, USA

Correspondence should be addressed to Michael J. Spurr; spurrm@ecu.edu

Received 23 February 2022; Accepted 28 March 2022; Published 7 July 2022

Academic Editor: Victor Kovtunenko

For a wide class of solutions to multiplicatively advanced differential equations (MADEs), a comprehensive set of relations is established between their Fourier transforms and Jacobi theta functions. In demonstrating this set of relations, the current study forges a systematic connection between the theory of MADEs and that of special functions. In a large subset of the general case, we introduce a new family of Schwartz wavelet MADE solutions $\mathscr{W}_{\mu,\lambda}(t)$ for μ and λ rational with $\lambda > 0$. These $\mathscr{W}_{\mu,\lambda}(t)$ have all moments vanishing and have a Fourier transform related to theta functions. For low parameter values derived from λ, the connection of the $\mathscr{W}_{\mu,\lambda}(t)$ to the theory of wavelet frames is begun. For a second set of low parameter values derived from λ, the notion of a canonical extension is introduced. A number of examples are discussed. The study of convergence of the MADE solution to the solution of its analogous ODE is begun via an in depth analysis of a normalized example $\mathscr{W}_{-4/3,1/3}(t)/\mathscr{W}_{-4/3,1/3}(0)$. A useful set of generalized q-Wallis formulas are developed that play a key role in this study of convergence.

1. Introduction

This paper expands the study of a class of solutions of multiplicatively advanced differential equations (MADEs) by determining the relationship of their Fourier transforms to Jacobi theta functions. The class of solutions under consideration consists of the Dirichlet-like [1] series:

$$f_{\mu,\lambda}(t) \equiv \sum_{m=-\infty}^{\infty} (-1)^m \frac{e^{-q^m t}}{q^{m(m-\mu)/\lambda}}, \quad \text{for } t \geq 0, \text{ where } \mu \in \mathbb{Q}, \lambda \in \mathbb{Q}^+ \text{ and } q > 1. \tag{1}$$

Each of the $f_{\mu,\lambda}(t)$ in (1) satisfies the MADE.

$$f_{\mu,\lambda}^{(\delta)}(t) = (-1)^{\gamma+\delta} q^{\gamma(\gamma+\mu)/\lambda} f_{\mu,\lambda}(q^\gamma t), \tag{2}$$

where $\lambda/2 = \gamma/\delta$ with γ/δ in a reduced form and $\gamma, \delta \in \mathbb{N}$; see [2]. Note that (2) is multiplicatively advanced in that the argument $q^\gamma t$ is an advancing of the parameter t by $q^\gamma > 1$, as q is taken to be greater than 1 and $\gamma \in \mathbb{N}$.

In general, as is shown in [2], there are nonunique ways to extend the $f_{\mu,\lambda}(t)$ from $0 \leq t < \infty$ to the negative reals, obtaining $F_{\mu,\lambda}(t)$ on $-\infty < t < \infty$ with $F_{\mu,\lambda}(t)$ also satisfying the MADE (2). To overcome this issue of nonuniqueness, we first extend $f_{\mu,\lambda}(t)$ (perhaps discontinuously) to the negative reals by defining $\tilde{f}_{\mu,\lambda}(t)$ to be.

$$\tilde{f}_{\mu,\lambda}(t) \equiv \begin{cases} f_{\mu,\lambda}(t), & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{3}$$

Then, we utilize only $\tilde{f}_{\mu,\lambda}(t)$ to obtain a new function $\mathscr{W}_{\mu,\lambda}(t)$ in $\mathscr{C}^\infty(\mathbb{R})$ which is naturally generated by $f_{\mu,\lambda}(t)$, and we further observe that for low parameters (namely, $1 \leq \delta \leq 3$ and $\delta = 0 \bmod \beta$) $\mathscr{W}_{\mu,\lambda}(t)$ is indeed an extension of $f_{\mu,\lambda}(t)$ satisfying (2). Here, β is obtained as $(\mu+1)/2 = \alpha/\beta$ with α/β in a reduced form.

In attempting to compute the Fourier transform

$$\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixt}\tilde{f}_{\mu,\lambda}(t)\,dt, \tag{4}$$

of $\tilde{f}_{\mu,\lambda}(t)$, we are led, instead, into discovering a relation between a weighted average of the $\mathcal{F}[\tilde{f}_{\mu,\lambda}(t)](x)$ of form

$$\frac{1}{M}\sum_{\ell=0}^{M-1}\left[\omega^{\ell}\right]^{p_1}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{[\omega^{\ell}]^{p_2}}\right), \tag{5}$$

where M, p_1, p_2 depend on μ, λ and ω is an M^{th} root of unity to a similar weighted average of $z_3^{p_3}/\theta(Q; z_3^M)$.

$$\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{p_3}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^{M}\right)}, \tag{6}$$

where M, D, γ, p_3 are determined by μ, λ and $\tilde{\omega}$ is a $(D\gamma)^{\text{th}}$ root of unity, while θ is the Jacobi theta function given by (10) below, z_3 is a scaled $(D\gamma)^{\text{th}}$ root of ix, and $Q=q^{2/\lambda}$. See equations (98) and (99) of Theorem 13 for further details.

One then applies the inverse Fourier transform to (5) to obtain the new functions

$$\mathcal{W}_{\mu,\lambda}(t)=\mathcal{F}^{-1}\left[\sum_{\ell=0}^{M-1}\left[\omega^{\ell}\right]^{p_1}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{[\omega^{\ell}]^{p_2}}\right)\right](t), \tag{7}$$

which are central to this study. These new $\mathcal{W}_{\mu,\lambda}(t)$ comprehensively generalize each of the main examples we have previously studied, including $K(q,t)$ in [3] and ${}_q\mathrm{Cos}(t)$, ${}_q\mathrm{Sin}(t)$ in [4, 2]. For certain low values of δ, we show that the $\mathcal{W}_{\mu,\lambda}(t)$ give unique, canonical extensions of the associated $f_{\mu,\lambda}(t)$ which satisfy the MADE (2). In general, for higher values of δ, the $\mathcal{W}_{\mu,\lambda}(t)$ are not extensions of $f_{\mu,\lambda}(t)$; however, they are all Schwartz wavelets with connections not only to wavelet theory but also to special function theory in that $\mathcal{F}[\mathcal{W}_{\mu,\lambda}(t)](x)$ is expressed in terms of (6) and thus in terms of the Jacobi theta function. See (98) and (99) of Theorem 13 for specifics.

We next determine a simple criteria for the $\mathcal{W}_{\mu,\lambda}(t)$ to not identically vanish, whereby the relationship of their Fourier transforms to the theta function remains substantive, as described in Theorem 23. This relationship is seen to greatly extend and generalize each of the special cases of Fourier transform computations that have been computed in all of our previous collected work, including [2–6]. Furthermore, even in those cases where $\mathcal{W}_{\mu,\lambda}(t)$ vanishes identically, we are still able to provide a (more technical) relation between the Fourier transform of $\tilde{f}_{\mu,\lambda}(t)$ and the Jacobi theta function in Theorem 28. We then begin applying the above discoveries to wavelet theory, producing numerous examples of new Schwartz wavelets generating frames for $\mathcal{L}^2(\mathbb{R})$.

Furthermore, considering the MADE (2) to be a perturbation of the classical ODE

$$f^{(\delta)}(t)=(-1)^{\gamma+\delta}f(t), \tag{8}$$

with $q>1$ in (2) considered to be the perturbation parameter as $q\longrightarrow 1^{+}$, we then initiate a study of convergence of the normalized solution of (2) to the classical solution of (8). In particular, we prove convergence for the normalized $\mathcal{W}_{-4/3,1/3}(t)$ to the classical solution of (8) with convergence being uniform on any compact set of $\mathbb{R}$; see Figure 1. We also exhibit graphical evidence for such convergence of other normalized $\mathcal{W}_{\mu,\lambda}(t)$.

The convergence seen in Figure 1 mirrors earlier convergence results [4] we have been previously able to obtain in the canonical extensions for the special cases: (1) $\mu=0$; $\lambda=1$ which normalizes to ${}_q\mathrm{Cos}(t)$ discussed in Section 5; and (2) $\mu=1$; $\lambda=1$ which normalizes to ${}_q\mathrm{Sin}(t)$, also discussed in Section 5. The convergence of ${}_q\mathrm{Cos}(t)$ to $\cos(t)$ and the convergence of ${}_q\mathrm{Sin}(t)$ to $\sin(t)$ are illustrated in Figure 2.

We conclude the paper with a set of generalizations of Wallis' formula for $\pi/2$ that we call generalized q-Wallis formulas, and we demonstrate their utility in the study of convergence of normalized solutions of MADEs to their classical analogue ODEs.

We mention that the current work falls in the area of functional differential equations of multiplicatively advanced type. Studies in functional differential equations include for instance [7–9]. More precisely, the current work falls under the area of q-difference differential equations, where the multiplicative advancement $y(t)\longrightarrow y(qt)$ is seen as a dilation that is denoted $\sigma_q[y](t)=y(qt)$. There is a robust study within the area of q-difference differential equations with dilations involving $q>1$. This is highlighted by works of L. Di Vizio [10–12]; C. Hardouin [11]; T. Dreyfus [13, 14]; A. Lastra [14], [15–20], [21–23]; S. Malek [14], [15–20], [21–23], [24–27]; J. Sanz [21–23]; H. Tahara [28]; and C. Zhang [12, 29], along with further references by these researchers and others. Also, for good background references to the current work, consult [2–6, 30–34] (especially [2, 4]). These last references also exhibit a number of various applications of global solutions of MADEs.

1.1. Preliminaries and Salient Properties of the Jacobi Theta Function. We shall need to extend the definition of $f_{\mu,\lambda}(t)$ to the case that the argument is complex and lying in the right half plane.

Definition 1. Let $q>1$, $\mu,\lambda\in\mathbb{Q}$, with $\lambda>0$. Then for $t\geq 0$ and the function $f_{\mu,\lambda}(t)$ given by (1), one defines for $\mathcal{R}(z)\geq 0$ (that is, for the real part of $z\in\mathbb{C}$ nonnegative)

$$f_{\mu,\lambda}(z)\equiv\sum_{m=-\infty}^{\infty}(-1)^{m}\frac{e^{-q^{m}z}}{q^{m(m-\mu)/\lambda}}, \tag{9}$$

which is analytic for $\mathcal{R}(z)>0$.

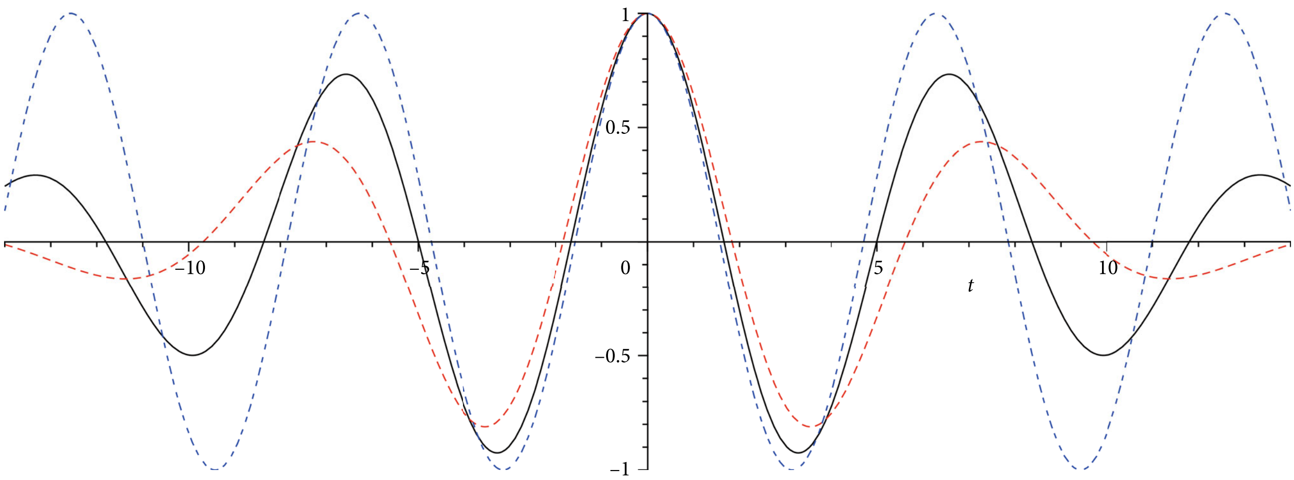
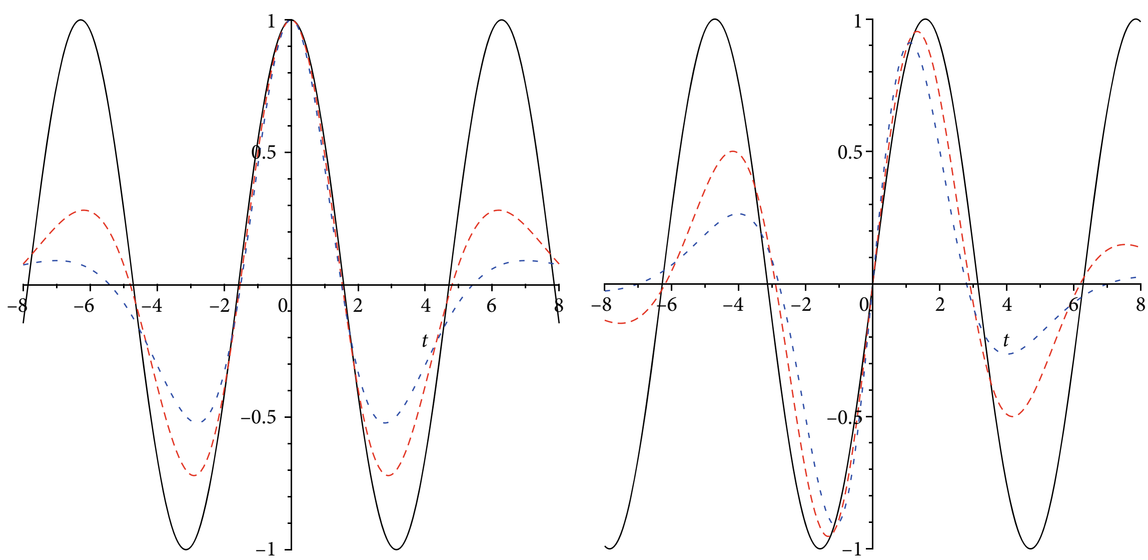


Figure 1: Plot of $y = \cos(t)$ (blue dots) approached by $y = \mathscr{W}_{-4/3,1/3}(t)/\mathscr{W}_{-4/3,1/3}(0)$ for: $q = 1.3$ (red dashed), $q = 1.1$ (solid black). Convergence is uniform on compact subsets of $\mathbb{R}$ as $q \longrightarrow 1^{+}$.



(a) (b)

Figure 2: (a) $y = \cos(x)$ (solid black) approached by $y = {}_q\mathrm{Cos}(t)$ for $q = 1.35$ (dotted-dash blue), $q = 1.15$ (dashed red). (b) $y = \sin(x)$ (solid black) approached by $y = {}_q\mathrm{Sin}(t)$ for $q = 1.35$ (dotted-dash blue), $q = 1.15$ (dashed red).

Next, recall that for $q > 1$ the Jacobi theta function is given by

$$\theta(q;u) = \sum_{n=-\infty}^{\infty} \frac{u^n}{q^{n(n-1)/2}} = \mu_q \prod_{n=0}^{\infty}\left(1 + \frac{u}{q^n}\right)\left(1 + \frac{1}{uq^{n+1}}\right), \tag{10}$$

where

$$\mu_q = \prod_{n=0}^{\infty}\left(1 - \frac{1}{q^{n+1}}\right). \tag{11}$$

Two properties of the Jacobi theta function of interest are that

$$\begin{gathered}\text{for all } p \in \mathbb{Z} \quad \theta(q;q^p u) = q^{p(p+1)/2} u^p \theta(q;u), \\ u\theta(q;u^{-1}) = \theta(q;u),\end{gathered} \tag{12}$$

which are proven in [5] and [6], respectively. From the product formula in (10), one sees that

$$\theta(q;u) = 0 \Longleftrightarrow u = -q^p \text{ for some } p \in \mathbb{Z}. \tag{13}$$

As indicated earlier, the Jacobi theta function plays a major role in this study in the computation of Fourier transforms.

2. Proof of the Relation of Fourier Transforms to Jacobi Theta Functions

We proceed immediately to the computation of Fourier transforms. For $\mu, \lambda \in \mathbb{R}$ with $\lambda > 0$ and given $f_{\mu,\lambda}(t)$ is as in (1), we define

$$\tilde{f}_{\mu,\lambda}(t) \equiv \begin{cases} f_{\mu,\lambda}(t) = \sum_{k=-\infty}^{\infty} (-1)^k \mathrm{e}^{-q^k t} / q^{k(k-\mu)/\lambda}, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases} \tag{14}$$

We now restrict $\mu, \lambda \in \mathbb{Q}$ to be rational with $\lambda > 0$, and let $x \in \mathbb{R}$. One then has the following computation of the Fourier transform $\mathscr{F}[\tilde{f}_{\mu,\lambda}(t)](x)$ of $\tilde{f}_{\mu,\lambda}(t)$:

$$\begin{aligned}
\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} \tilde{f}_{\mu,\lambda}(t) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ixt} \sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-q^k t}}{q^{k(k-\mu)/\lambda}} dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{q^{k(k-\mu)/\lambda}} \int_{0}^{\infty} e^{-ixt - q^k t} dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{q^{k(k-\mu)/\lambda}} \frac{e^{-ixt-q^k t}}{-ix - q^k} \Bigg|_{0}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{1}{(ix + q^k)} \right] \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k+1)/\lambda}} \frac{q^{k(\mu+1)/\lambda}}{(ix + q^k)} \right] \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{\left(q^{2/\lambda}\right)^{k(k+1)/2}} \frac{\left(q^{2/\lambda}\right)^{k(\mu+1)/2}}{\left(ix + \left(q^{2/\lambda}\right)^{k\lambda/2}\right)} \right] \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{Q^{k(\mu+1)/2}}{\left(ix + Q^{k\lambda/2}\right)} \right]
\end{aligned} \tag{15}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{\left[Q^k\right]^{(\alpha/\beta)}}{\left(ix + \left[Q^k\right]^{(\gamma/\delta)}\right)} \right], \tag{16}$$

where in (15) and (16) $Q \equiv q^{2/\lambda}$ and for conciseness in moving from (15) to (16), one has $(\mu + 1)/2 \equiv \alpha/\beta$ and $\lambda/2 \equiv \gamma/\delta$, where α/β and γ/δ in $\mathbb{Q}$ are taken to be in a reduced form with $\alpha \in \mathbb{Z}$ and $\beta, \gamma, \delta \in \mathbb{N}$. We extend $\mathscr{F}[\tilde{f}_{\mu,\lambda}(t)](x)$ to the complex plane by setting

$$\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](\zeta) \equiv \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{\left[Q^k\right]^{(\alpha/\beta)}}{\left(i\zeta + \left[Q^k\right]^{(\gamma/\delta)}\right)} \right], \tag{17}$$

for $\zeta \in \mathbb{C} \setminus S$ where $S = \{iQ^{k\gamma/\delta} \mid k \in \mathbb{Z}\}$ denotes the set of ζ where the denominator in (17) vanishes. Note that $\mathscr{F}[\tilde{f}_{\mu,\lambda}(t)](\zeta)$ is defined at $\zeta = 0$ by virtue of the quadratic exponent $k(k+1)/2$ of Q in the denominator counteracting any growth of the linear exponent $k[\alpha/\beta - \gamma/\delta]$ of Q in the numerator of (17). Also for $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, note that, if U is any open region with compact closure $\bar{U} \subset \mathbb{C}^* \setminus S$, one has that for $\zeta \in U$ the distance d_ζ from ζ to S is positive, as the only cluster point of S is 0. Furthermore, the distance $d(\bar{U}, S)$ from $\bar{U}$ to S is also positive. Hence, we have

$$\begin{aligned}
&\left| \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{\left[Q^k\right]^{(\alpha/\beta)}}{\left(i\zeta + \left[Q^k\right]^{(\gamma/\delta)}\right)} \right] \right| \\
&\quad \leq \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{1}{Q^{k(k+1)/2}} \frac{\left[Q^k\right]^{(\alpha/\beta)}}{d_\zeta} \right] \\
&\quad \leq \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{1}{Q^{k(k+1)/2}} \frac{\left[Q^k\right]^{(\alpha/\beta)}}{d\left(\bar{U}, S\right)} \right] < \infty.
\end{aligned} \tag{18}$$

Hence, the truncated sums

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-N}^{N} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{\left[Q^k\right]^{(\alpha/\beta)}}{\left(i\zeta + \left[Q^k\right]^{(\gamma/\delta)}\right)} \right], \tag{19}$$

which are analytic on U, approach $\mathscr{F}[\tilde{f}_{\mu,\lambda}(t)](\zeta)$ uniformly on U as $N \longrightarrow \infty$. Thus, $\mathscr{F}[\tilde{f}_{\mu,\lambda}(t)](\zeta)$ is analytic on U [35] and therefore analytic on $\mathbb{C}^* \setminus S$.

We have seen in [2] that the "alternating Q-combinatoric" $(-1)^k / Q^{k(k+1)/2}$ in (16) can be given by the residue of $1/[u\theta(Q; u)]$ at a simple pole $u = -Q^k$ under a computation of an appropriate contour integral about a region containing $u = -Q^k$. Here, $\theta(Q; u)$ is the Jacobi theta function given by (10). Observe that the term $[Q^k]^{(\alpha/\beta)} / (ix + [Q^k]^{(\gamma/\delta)})$ in (16) would then be obtained from evaluation of $[-u]^{(\alpha/\beta)} / (ix + [-u]^{\gamma/\delta})$ at $u = -Q^k$. Therefore, as a starting point, we would be interested in integrating the expression

$$\frac{1}{u\,\theta(Q; u)} \frac{[-u]^{\alpha/\beta}}{\left(z + [-u]^{\gamma/\delta}\right)} du, \tag{20}$$

around an appropriate closed contour Γ in the complex u-plane, where we set $z = ix$ later. However, since there is in general a multivalued issue with expression (20) if α/β or γ/δ are not integers, we set $u = v^M$ in (20), where $M > 0$ is the least integer such that $M\alpha/\beta$ and $M\gamma/\delta$ are both integers, and we integrate

$$\int_{\Gamma} \frac{1}{\nu^M \theta(Q;\nu^M)} \frac{[-\nu^M]^{(\alpha/\beta)}}{\left(z+[-\nu^M]^{(\gamma/\delta)}\right)} M\nu^{M-1} d\nu = M\int_{\Gamma} \frac{1}{\nu\,\theta(Q;\nu^M)} \frac{[\nu^\alpha]^{(M/\beta)} e^{i\pi\alpha/\beta}}{\left(z+[\nu^\gamma]^{(M/\delta)} e^{i\pi\gamma/\delta}\right)} d\nu. \tag{21}$$

Note that, since α and β have no common factors and since $M\alpha/\beta \in \mathbb{Z}$, one has that β divides M. Similarly, since γ and δ have no common factors and since $M\gamma/\delta \in \mathbb{N}$, δ also divides M. Since M is the least such integer, M is the least common multiple of β and δ. Thus,

$$\text{for } \alpha \in \mathbb{Z}, \quad \beta,\gamma,\delta \in \mathbb{N} \text{ with } \frac{\mu+1}{2} = \frac{\alpha}{\beta}, \quad \frac{\lambda}{2} = \frac{\gamma}{\delta} \text{ in reduced form,} \tag{22}$$

we set

$$M = \operatorname{lcm}\{\beta,\delta\}, \quad B = \frac{M}{\beta}, \quad D = \frac{M}{\delta}. \tag{23}$$

We then divide (21) by M and integrate the following simplified version of (21)

$$\int_{\Gamma} \frac{1}{\nu\,\theta(Q;\nu^M)} \frac{\nu^{B\alpha} e^{i\pi\alpha/\beta}}{\left(z+\nu^{D\gamma} e^{i\pi\gamma/\delta}\right)} d\nu, \tag{24}$$

around a closed contour Γ in the complex ν-plane, where the exponents M, $B\alpha$, and $D\gamma$ are now integers (avoiding any multivalued issue in a would-be contour integral involving (20) by instead using the integration in (24)). The contour Γ will later be taken to be the oriented boundary of an annulus centered at the origin. This key step in avoiding multivalued issues in moving away from (20) to the integral in (24) allows us to overcome the limiting assumptions in earlier work (that μ is odd and λ is even in Theorem 6.3 of [2] or that μ is an integer and λ is 1 in Theorem 6.5 of [2]) to now handle the general case in this study (where μ and λ are allowed to be rational, with $\lambda > 0$).

In anticipation of a residue computation of the expression (24), we begin by examining the product representation of the Jacobi theta function $\theta(Q;u)$ in (10) and removing one appropriate factor from the product corresponding to the vanishing of $\theta(Q;u)$ when $u = -Q^k$. That is, note that from (10), one has that for $k \geq 0$

$$\begin{aligned}\theta(Q;u) &= \mu_Q \prod_{n=0}^{\infty}\left(1+\frac{u}{Q^n}\right)\left(1+\frac{1}{uQ^{n+1}}\right) \\ &= \left(1+\frac{u}{Q^k}\right)\left[\mu_Q \prod_{n=0,n\neq k}^{\infty}\left(1+\frac{u}{Q^n}\right)\prod_{n=0}^{\infty}\left(1+\frac{1}{uQ^{n+1}}\right)\right] \end{aligned}\tag{25}$$

$$= \left(1+\frac{u}{Q^k}\right)\theta(k\mid Q;u) = \left(\frac{Q^k+u}{Q^k}\right)\theta(k\mid Q;u), \tag{26}$$

where for $k \geq 0$, the expression $\theta(k\mid Q;u)$ in (26) is defined by the bracketed expression in (25), namely,

$$\theta(k\mid Q;u) \equiv \left[\mu_Q \prod_{n=0,n\neq k}^{\infty}\left(1+\frac{u}{Q^n}\right)\prod_{n=0}^{\infty}\left(1+\frac{1}{uQ^{n+1}}\right)\right]. \tag{27}$$

Similarly, for $k < 0$ one has that

$$\begin{aligned}\theta(Q;u) &= \mu_Q \prod_{n=0}^{\infty}\left(1+\frac{u}{Q^n}\right)\left(1+\frac{1}{uQ^{n+1}}\right) \\ &= \left(1+\frac{1}{uQ^{|k|}}\right)\left[\mu_Q \prod_{n=0}^{\infty}\left(1+\frac{u}{Q^n}\right)\prod_{n=0,n\neq|k|-1}^{\infty}\left(1+\frac{1}{uQ^{n+1}}\right)\right] \end{aligned}\tag{28}$$

$$= \left(1+\frac{1}{uQ^{|k|}}\right)\theta(k\mid Q;u) = \left(\frac{u+Q^k}{u}\right)\theta(k\mid Q;u), \tag{29}$$

where for $k < 0$, the expression $\theta(k\mid Q;u)$ in (29) is defined by the bracketed expression in (28), namely,

$$\theta(k\mid Q;u) \equiv \left[\mu_Q \prod_{n=0}^{\infty}\left(1+\frac{u}{Q^n}\right)\prod_{n=0,n\neq|k|-1}^{\infty}\left(1+\frac{1}{uQ^{n+1}}\right)\right]. \tag{30}$$

Thus, via (27) and (30), the expression $\theta(k\mid Q;u)$ is defined for each $k \in \mathbb{Z}$.

We pause the discussion on representing $\theta(Q;u)$ in terms of $\theta(k\mid Q;u)$ in order to record a series of useful computational lemmas. The first such lemma evaluates $\theta(k\mid Q;-Q^k)$ as an "alternating combinatoric."

Lemma 2. *For $k \geq 0$ and $\theta(k\mid Q;u)$ as in (26), one has*

$$\theta\left(k\mid Q;-Q^k\right) = \left[\mu_Q \prod_{n=0,n\neq k}^{\infty}\left(1-\frac{Q^k}{Q^n}\right)\prod_{n=0}^{\infty}\left(1-\frac{1}{Q^{k+n+1}}\right)\right] = (-1)^k \mu_Q^3 Q^{k(k+1)/2}. \tag{31}$$

And for $k < 0$ and $\theta(k \mid Q; u)$ as in (29), one has

$$\theta\left(k \mid Q; -Q^k\right) = \left[\mu_Q \prod_{n=0}^{\infty}\left(1 - \frac{Q^k}{Q^n}\right) \prod_{n=0, n\neq|k|-1}^{\infty}\left(1 - \frac{1}{Q^{k+n+1}}\right)\right] = -(-1)^k \mu_Q^3 Q^{k(k+1)/2}. \quad (32)$$

Proof. The proof is given in Lemma 5.1 of [2]. □

The second lemma will provide a structure for the proof of the third lemma, and it will be utilized in a subsequent residue computation.

Lemma 3. *For an integer $M \geq 2$, let $\omega = e^{2\pi i/M}$ be an M^{th} root of unity. Then,*

$$x^M - b^M = (x-b)\sum_{j=0}^{M-1} x^j b^{M-1-j} = (x-b)\prod_{p=1}^{M-1}(x - \omega^p b). \quad (33)$$

Hence,

$$\sum_{j=0}^{M-1} x^j b^{M-1-j} = \prod_{p=1}^{M-1}(x - \omega^p b). \quad (34)$$

Proof. Upon expansion of the middle expression in (33), the left-most equality is self-evident. The right-most equality in (33) follows from the fact that for each $p \in \{0, 1, \cdots, M-1\}$ one has $\omega^p b$ is a root of $x^M - b^M$, and hence, $x - \omega^p b$ is a factor. To obtain (34), one divides the right two expressions in (33) by $(x - b)$. The lemma is now proven. □

The third lemma will simplify the computation in (50) below.

Lemma 4. *For an integer $M \geq 2$ let $\omega = e^{2\pi i/M}$ be an M^{th} root of unity. One has*

$$\prod_{p=1}^{M-1}(1 - \omega^p) = M. \quad (35)$$

Proof. Set $x = 1 = b$ in (34) to obtain (35). The lemma is shown. □

We record three further lemmas on the behavior of roots of unity for later computational use.

Lemma 5. *Let $\omega = e^{2\pi i/M}$ be an M^{th} root of unity, and let $\phi = \omega^p$ for some $p \in \mathbb{Z}$ have order N. Then,*

$$\sum_{\ell=0}^{N-1}\phi^\ell = \begin{cases} 1, & \text{if } N = 1, \\ 0, & \text{if } N \geq 2, \end{cases} \quad \text{In particular,} \quad \sum_{\ell=0}^{M-1}\omega^\ell = \begin{cases} 1, & \text{if } M = 1, \\ 0, & \text{if } M \geq 2. \end{cases} \quad (36)$$

Proof. If $N = 1$, then $\phi^0 = 1$. If $N \geq 2$, then let $\sigma = \sum_{\ell=0}^{N-1}\phi^\ell$. Observe that since $\phi^N = \phi^0$ we have $\phi\sigma = \sigma$ which gives $0 = (1-\phi)\sigma$. Since $1 - \phi \neq 0$, we conclude $\sigma = 0$. In particular, if $p = 1$, then $N = M$ and the second equality in (36) holds. The lemma is demonstrated. □

The next lemma generalizes the previous lemma.

Lemma 6. *Let $\omega = e^{2\pi i/M}$ be an M^{th} root of unity. Let $p \in \mathbb{Z}$ be fixed. Then,*

$$\sum_{\ell=0}^{M-1}\left[\omega^\ell\right]^p = \begin{cases} M, & \text{if } p = 0 \mod M, \\ 0, & \text{if } p \neq 0 \mod M. \end{cases} \quad (37)$$

Proof. If p is a multiple of M then $[\omega^\ell]^p = 1$ and then $\sum_{\ell=0}^{M-1}[\omega^\ell]^p = \sum_{\ell=0}^{M-1} 1 = M$. If p is not divisible by M, then $\omega^p \neq 1$ is a root of unity with order, say, $N > 1$, with $NT = M$. Then,

$$\sum_{\ell=0}^{M-1}\left[\omega^\ell\right]^p = \sum_{\ell=0}^{M-1}[\omega^p]^\ell = \sum_{\ell=0}^{N-1}[\omega^p]^\ell + \sum_{\ell=N}^{2N-1}[\omega^p]^\ell + \cdots \sum_{\ell=(j-1)N}^{jN-1}[\omega^p]^\ell + \cdots + \sum_{\ell=(T-1)N}^{TN-1}[\omega^p]^\ell \quad (38)$$

$$= 0, \quad (39)$$

where the vanishing in (39) follows from the vanishing of each summand

$$\sum_{\ell=(j-1)N}^{jN-1}[\omega^p]^\ell = [\omega^p]^{(j-1)N}\sum_{\ell=0}^{N-1}[\omega^p]^\ell, \quad (40)$$

in (38), which in turn follows from an application of Lemma 5. This proves the lemma. □

The following is a refinement of Lemma 6.

Lemma 7. *Let $\omega = e^{2\pi i/M}$ be an M^{th} root of unity, with $M = ab$, where $a, b \in \mathbb{N}$. Let $p \in \mathbb{Z}$ be fixed. Then if $a = 1$, one has*

$$\sum_{\ell=0}^{a-1}\left[\omega^\ell\right]^p = 1, \quad \text{for all } p, \quad (41)$$

and if $a > 1$, one has

$$\sum_{\ell=0}^{a-1}\left[\omega^{\ell}\right]^{p}=\begin{cases} a, & \text{if } p=0 \bmod M,\\ 0, & \text{if } p\neq 0 \bmod M \text{ and } p=0 \bmod b,\\ \dfrac{\left([\omega^{a}]^{p}-1\right)}{\left(\omega^{p}-1\right)}\neq 0, & \text{if } p\neq 0 \bmod M \text{ and } p\neq 0 \bmod b. \end{cases} \quad (42)$$

Proof. Let $\sigma=\sum_{\ell=0}^{a-1}[\omega^{\ell}]^{p}$. Then, if $a=1$, one has $\sigma=[\omega^{0}]^{p}=1^{p}=1$, giving (41). If $a>1$, one has

$$\omega^{p}\sigma=\sum_{\ell=0}^{a-1}\left[\omega^{\ell+1}\right]^{p}=[\omega^{a}]^{p}+\sum_{\ell=0}^{a-1}\left[\omega^{\ell}\right]^{p}-\left[\omega^{0}\right]^{p}=[\omega^{a}]^{p}+\sigma-1. \quad (43)$$

Hence,

$$[\omega^{p}-1]\sigma=[\omega^{a}]^{p}-1. \quad (44)$$

If $p=0 \bmod M$, $\sigma=\sum_{\ell=0}^{a-1}[\omega^{\ell}]^{p}=\sum_{\ell=0}^{a-1}1=a$, giving the first case in (42). If $p\neq 0 \bmod M$ and $p=0 \bmod b$, then ω^{a} is a b^{th} root of unity and the right hand side of (44) vanishes while $[\omega^{p}-1]\neq 0$, giving $\sigma=0$ for the second case. Finally, if $p\neq 0 \bmod M$ and $p\neq 0 \bmod b$, then ω^{a} is a b^{th} root of unity and the right hand side of (44) does not vanish while $[\omega^{p}-1]\neq 0$, resulting in $\sigma=([\omega^{a}]^{p}-1)/[\omega^{p}-1]\neq 0$. This shows the third case and finishes the proof. Note that the third case is the only case with $p\neq 0 \bmod b$, because in the first case if $p=0 \bmod M$ with $M=ab$, then $p=0 \bmod b$. This gives the lemma. □

We return now to the discussion in (25)–(30). Let $k\geq 0$. In (25) and (26), we let $u=\nu^{M}$ and $\omega=e^{2\pi i/M}$ be the M^{th} root of unity; one then has that, for each ℓ with $0\leq\ell\leq M-1$,

$$\begin{aligned}\theta\left(Q;\nu^{M}\right)&=\mu_{Q}\prod_{n=0}^{\infty}\left(1+\frac{\nu^{M}}{Q^{n}}\right)\left(1+\frac{1}{\nu^{M}Q^{n+1}}\right)\\&=\left(1+\frac{\nu^{M}}{Q^{k}}\right)\left[\mu_{Q}\prod_{n=0,n\neq k}^{\infty}\left(1+\frac{\nu^{M}}{Q^{n}}\right)\prod_{n=0}^{\infty}\left(1+\frac{1}{\nu^{M}Q^{n+1}}\right)\right]\end{aligned} \quad (45)$$

$$=\left(1+\frac{\nu^{M}}{Q^{k}}\right)\theta\left(k\mid Q;\nu^{M}\right)=\left(\frac{Q^{k}+\nu^{M}}{Q^{k}}\right)\theta\left(k\mid Q;\nu^{M}\right) \quad (46)$$

$$\begin{aligned}&=\left(\frac{\prod_{j=0}^{M-1}\left(\nu-\omega^{j}e^{i\pi/M}Q^{k/M}\right)}{Q^{k}}\right)\theta\left(k\mid Q;\nu^{M}\right)\\&=\frac{\left(\nu-\omega^{\ell}e^{i\pi/M}Q^{k/M}\right)}{Q^{k}}\left(\prod_{j=0,j\neq\ell}^{M-1}\left(\nu-\omega^{j}e^{i\pi/M}Q^{k/M}\right)\right)\cdot\theta\left(k\mid Q;\nu^{M}\right),\end{aligned} \quad (47)$$

where (27) was used to move from (45) to (46) and (33) in Lemma 3 was used to move from (46) to (47). Thus, for $k\geq 0$, the residue of $f_{1}(\nu)=1/[\nu\theta(Q;\nu^{M})]$ at $\nu_{k,\ell}=\omega^{\ell}e^{i\pi/M}Q^{k/M}$ is given by

$$\operatorname{Res}\left(f_{1},\nu_{k,\ell}\right)=\frac{Q^{k}}{\omega^{\ell}e^{i\pi/M}Q^{k/M}}\cdot\frac{1}{\left(\prod_{j=0,j\neq\ell}^{M-1}\left(\omega^{\ell}e^{i\pi/M}Q^{k/M}-\omega^{j}e^{i\pi/M}Q^{k/M}\right)\right)}\cdot\frac{1}{\theta\left(k\mid Q;\left[\omega^{\ell}e^{i\pi/M}Q^{k/M}\right]^{M}\right)} \quad (48)$$

$$=\frac{Q^{k}}{\left[\omega^{\ell}e^{i\pi/M}Q^{k/M}\right]^{M}}\frac{1}{\left(\prod_{j=0,j\neq\ell}^{M-1}\left(1-\omega^{j-\ell}\right)\right)\theta\left(k\mid Q;-Q^{k}\right)} \quad (49)$$

$$=\frac{Q^{k}}{-Q^{k}}\frac{1}{M\left[\theta\left(k\mid Q;-Q^{k}\right)\right]}=\frac{1}{-1}\frac{1}{M\left[(-1)^{k}\mu_{Q}^{3}Q^{k(k+1)/2}\right]} \quad (50)$$

$$=\frac{(-1)^{k+1}}{\mu_{Q}^{3}Q^{k(k+1)/2}}\frac{1}{M}, \quad (51)$$

where we have factored out $\omega^{\ell}e^{i\pi/M}Q^{k/M}$ from each factor in $(\prod_{j=0,j\neq\ell}^{M-1}(\omega^{\ell}e^{i\pi/M}Q^{k/M}-\omega^{j}e^{i\pi/M}Q^{k/M}))$ in the denominator of (48) to obtain (49); and the first equality in (50) follows from Lemma 4 if $M\geq 2$ (and is automatic if $M=1$); and the second equality in (50) follows from (31) of Lemma 2.

Let $k<0$. In (28) and (29), we let $u=\nu^{M}$ and $\omega=e^{2\pi i/M}$; one then has that, for each ℓ with $0\leq\ell\leq M-1$,

$$\begin{aligned}\theta\left(Q;\nu^{M}\right)&=\mu_{Q}\prod_{n=0}^{\infty}\left(1+\frac{\nu^{M}}{Q^{n}}\right)\left(1+\frac{1}{\nu^{M}Q^{n+1}}\right)\\&=\left(1+\frac{1}{\nu^{M}Q^{|k|}}\right)\left[\mu_{Q}\prod_{n=0}^{\infty}\left(1+\frac{\nu^{M}}{Q^{n}}\right)\prod_{n=0,n\neq|k|-1}^{\infty}\left(1+\frac{1}{\nu^{M}Q^{n+1}}\right)\right]\end{aligned} \quad (52)$$

$$=\left(1+\frac{1}{\nu^{M}Q^{|k|}}\right)\theta\left(k\mid Q;\nu^{M}\right)=\left(\frac{\nu^{M}+Q^{k}}{\nu^{M}}\right)\theta\left(k\mid Q;\nu^{M}\right) \quad (53)$$

$$\begin{aligned}&=\left(\frac{\prod_{j=0}^{M-1}\left(\nu-\omega^{j}e^{i\pi/M}Q^{k/M}\right)}{\nu^{M}}\right)\theta\left(k\mid Q;\nu^{M}\right)\\&=\frac{\left(\nu-\omega^{\ell}e^{i\pi/M}Q^{k/M}\right)}{\nu^{M}}\left(\prod_{j=0,j\neq\ell}^{M-1}\left(\nu-\omega^{j}e^{i\pi/M}Q^{k/M}\right)\right)\cdot\theta\left(k\mid Q;\nu^{M}\right),\end{aligned} \quad (54)$$

where (30) was used to move from (52) to (53) and (33) in Lemma 3 was used to move from (53) to (54). Thus, for $k < 0$, the residue of $f_1(\nu) = 1/[\nu\theta(Q;\nu^M)]$ at $\nu_{k,\ell} = \omega^\ell e^{i\pi/M} Q^{k/M}$ is given by

$$\operatorname{Res}(f_1, \nu_{k,\ell}) = \frac{\left[\omega^\ell e^{i\pi/M} Q^{k/M}\right]^M}{\omega^\ell e^{i\pi/M} Q^{k/M}} \cdot \frac{1}{\left(\prod_{j=0,j\neq\ell}^{M-1}\left(\omega^\ell e^{i\pi/M} Q^{k/M} - \omega^j e^{i\pi/M} Q^{k/M}\right)\right)} \cdot \frac{1}{\theta\left(k \mid Q; \left[\omega^\ell e^{i\pi/M} Q^{k/M}\right]^M\right)} \tag{55}$$

$$= \frac{-Q^k}{\left[\omega^\ell e^{i\pi/M} Q^{k/M}\right]^M} \frac{1}{\left(\prod_{j=0,j\neq\ell}^{M-1}\left(1-\omega^{j-\ell}\right)\right)\theta\left(k \mid Q; -Q^k\right)} \tag{56}$$

$$= \frac{-Q^k}{-Q^k} \frac{1}{M\left[\theta\left(k \mid Q; -Q^k\right)\right]} = \frac{1}{1} \frac{1}{M\left[-(-1)^k \mu_Q^3 Q^{k(k+1)/2}\right]} \tag{57}$$

$$= \frac{(-1)^{k+1}}{\mu_Q^3 Q^{k(k+1)/2}} \frac{1}{M}, \tag{58}$$

where we have factored out $\omega^\ell e^{i\pi/M} Q^{k/M}$ from each factor in $(\prod_{j=0,j\neq\ell}^{M-1}(\omega^\ell e^{i\pi/M} Q^{k/M} - \omega^j e^{i\pi/M} Q^{k/M}))$ in the denominator of (55) to obtain (56); and the first equality in (57) follows from Lemma 4 if $M \geq 2$ (and is automatic if $M = 1$); and the second equality in (57) follows from (32) of Lemma 2.

Note that the form of the final expression in (51) agrees with the form of the final expression in (58). Hence, we have that for all $k \in \mathbb{Z}$ the residue of $f_1(\nu) = 1/[\nu\theta(Q;\nu^M)]$ at $\nu_{k,\ell} = \omega^\ell e^{i\pi/M} Q^{k/M}$ is given by

$$\operatorname{Res}(f_1, \nu_{k,\ell}) = \frac{(-1)^{k+1}}{\mu_Q^3 Q^{k(k+1)/2}} \frac{1}{M}. \tag{59}$$

The previous discussion allows us to reach the following conclusion:

Proposition 8. *Let $\alpha, \beta, \gamma, \delta, M, B$ and D be as in (22) and (23). Let $k \in \mathbb{Z}$, $\omega = e^{2\pi i/M}$, and ℓ satisfying $0 \leq \ell \leq M-1$ all be fixed. Let $z \in \mathbb{C}\backslash\{-[\omega^{\ell+1} Q^{k/M}]^{D\gamma}\}$ be fixed. Then the residue of*

$$f_2(\nu) = \frac{1}{\nu\theta(Q;\nu^M)} \frac{\nu^{B\alpha} e^{i\pi\alpha/\beta}}{\left(z + \nu^{D\gamma} e^{i\pi\gamma/\delta}\right)} \tag{60}$$

at $\nu_{k,\ell} = \omega^\ell e^{i\pi/M} Q^{k/M}$ is given by

$$Res(f_2, \nu_{k,\ell}) = \frac{(-1)^{k+1}}{\mu_Q^3 Q^{k(k+1)/2}} \frac{1}{M} \frac{\left[\omega^\ell e^{i\pi/M} Q^{k/M}\right]^{B\alpha} e^{i\pi\alpha/\beta}}{\left(z + \left[\omega^\ell e^{i\pi/M} Q^{k/M}\right]^{D\gamma} e^{i\pi\gamma/\delta}\right)} \tag{61}$$

$$= \frac{(-1)^{k+1}}{\mu_Q^3 Q^{k(k+1)/2}} \frac{1}{M} \frac{\left[\omega^{\ell+1} Q^{k/M}\right]^{B\alpha}}{\left(z + \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}\right)}. \tag{62}$$

Proof. Referring to (24) and (59) and the discussion above, equality in (61) follows immediately, once one determines that

$$\frac{\nu^{B\alpha} e^{i\pi\alpha/\beta}}{\left(z + \nu^{D\gamma} e^{i\pi\gamma/\delta}\right)} \tag{63}$$

is analytic in ν at $\nu_{k,\ell} = \omega^\ell e^{i\pi/M} Q^{k/M}$. Thus, we must require that $z + [\omega^\ell e^{i\pi/M} Q^{k/M}]^{D\gamma} e^{i\pi\gamma/\delta} \neq 0$, which is equivalent to $z \neq -[\omega^{\ell+1} Q^{k/M}]^{D\gamma}$, as seen in the next sentence. Equality in (62) follows directly from the facts that $e^{i\pi\alpha/\beta} = [e^{i\pi/M}]^{M\alpha/\beta} = [e^{i\pi/M}]^{B\alpha}$ and $e^{i\pi\gamma/\delta} = [e^{i\pi/M}]^{M\gamma/\delta} = [e^{i\pi/M}]^{D\gamma}$. The proposition is now shown. □

Note that when $\ell = M-1$ and $z = ix$ in (62), the resulting expression matches the k^{th} summand in (16) up to the constant factors $1/\sqrt{2\pi}$ and $(-1/[M\mu_Q^3])$.

Having found the residues at $\omega^\ell e^{i\pi/M} Q^{k/M}$, the next proposition allows for the determination of the residues of $f_2(\nu)$ in (60) at the roots of $(z + \nu^{D\gamma} e^{i\pi\gamma/\delta})$. First, observe that $(z + \nu^{D\gamma} e^{i\pi\gamma/\delta}) = 0$ precisely when $\nu^{D\gamma} = e^{i\pi} e^{-i\pi\gamma/\delta} z$, namely, when $\nu = \tilde{\omega}^\kappa e^{i\pi/[D\gamma]} e^{-i\pi\gamma/[\delta D\gamma]} z_0$ where z_0 is a $[D\gamma]^{\text{th}}$ root of z, $\tilde{\omega} = e^{2\pi i/[D\gamma]}$ is the $[D\gamma]^{\text{th}}$ root of unity, and $0 \leq \kappa \leq D\gamma - 1$.

Proposition 9. *Let $z \in \mathbb{C}^*\backslash\tilde{S}$ where $\tilde{S} \equiv \{-[\omega^{j+1} Q^{k/M}]^{D\gamma} \mid k \in \mathbb{Z}, 0 \leq j \leq M-1\}$. For $0 \leq \kappa \leq D\gamma - 1$, $\tilde{\omega} = e^{2\pi i/[D\gamma]}$, z_0 a fixed $[D\gamma]^{th}$ root of z, and $z_1 = e^{i\pi/[D\gamma]} e^{-i\pi\gamma/[\delta D\gamma]} z_0$, the residue of*

$$f_2(\nu) = \frac{1}{\nu\theta(Q;\nu^M)} \frac{\nu^{B\alpha} e^{i\pi\alpha/\beta}}{\left(z + \nu^{D\gamma} e^{i\pi\gamma/\delta}\right)} \tag{64}$$

at $\nu_\kappa = \tilde{\omega}^\kappa e^{i\pi/[D\gamma]} e^{-i\pi\gamma/[\delta D\gamma]} z_0 = \tilde{\omega}^\kappa z_1$ is given by

$$Res(f_2, \nu_\kappa) = \frac{1}{D\gamma} \frac{1}{[-z]} \frac{[\tilde{\omega}^\kappa z_1]^{B\alpha} e^{i\pi\alpha/\beta}}{\theta\left(Q; [\tilde{\omega}^\kappa z_1]^M\right)}. \tag{65}$$

Proof. Equation (65) holds for $D\gamma = 1$ upon setting $\nu = -e^{-i\pi\gamma/\delta} z$ in the expression $\nu^{B\alpha} e^{i\pi\alpha/\beta}/[\nu\theta(Q;\nu^M) e^{i\pi\gamma/\delta}]$. Observe that for $D\gamma \geq 2$ the expression $(z + \nu^{D\gamma} e^{i\pi\gamma/\delta})$ can be factored as follows:

$$\left(z+\nu^{D\gamma}e^{i\pi\gamma/\delta}\right)=e^{i\pi\gamma/\delta}\left(\nu^{D\gamma}-e^{i\pi}e^{-i\pi\gamma/\delta}z\right) \tag{66}$$

$$=e^{i\pi\gamma/\delta}\prod_{j=0}^{D\gamma-1}\left(\nu-\tilde{\omega}^{j}e^{i\pi/[D\gamma]}e^{-i\pi\gamma/[\delta D\gamma]}z_0\right) \tag{67}$$

$$=e^{i\pi\gamma/\delta}\prod_{j=0}^{D\gamma-1}\left(\nu-\tilde{\omega}^{j}z_1\right) \tag{68}$$

$$=\left(\nu-\tilde{\omega}^{\kappa}z_1\right)\left[e^{i\pi\gamma/\delta}\prod_{j=0,j\neq\kappa}^{D\gamma-1}\left(\nu-\tilde{\omega}^{j}z_1\right)\right], \tag{69}$$

where one uses (33) to move from (66) to (67) and one uses the definition $z_1=e^{i\pi/[D\gamma]}e^{-i\pi\gamma/[\delta D\gamma]}z_0$ to simplify (67) into (68). Notice next that evaluation of the bracketed expression in (69) at $\tilde{\omega}^{\kappa}z_1$ yields

$$\begin{aligned}&\left[e^{i\pi\gamma/\delta}\prod_{j=0,j\neq\kappa}^{D\gamma-1}\left(\tilde{\omega}^{\kappa}z_1-\tilde{\omega}^{j}z_1\right)\right]\\&=e^{i\pi\gamma/\delta}\left[\tilde{\omega}^{\kappa}z_1\right]^{D\gamma-1}\left[\prod_{j=0,j\neq\kappa}^{D\gamma-1}\left(1-\tilde{\omega}^{j-\kappa}\right)\right]\\&=e^{i\pi\gamma/\delta}\left[\tilde{\omega}^{\kappa}z_1\right]^{D\gamma-1}[D\gamma],\end{aligned} \tag{70}$$

where equality in (70) follows from (35) in Lemma 4. □

To compute the residue of $f_2(\nu)$ in (64) at $\nu=\nu_\kappa=\tilde{\omega}^{\kappa}z_1$ one observes

$$\begin{aligned}\operatorname{Res}(f_2,\nu_\kappa)=&\frac{1}{[\tilde{\omega}^{\kappa}z_1]\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^{M}\right)}\\&\cdot\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}e^{i\pi\alpha/\beta}}{\left[e^{i\pi\gamma/\delta}\prod_{j=0,j\neq\kappa}^{D\gamma-1}\left(\tilde{\omega}^{\kappa}z_1-\tilde{\omega}^{j}z_1\right)\right]}\end{aligned} \tag{71}$$

$$=\frac{1}{[\tilde{\omega}^{\kappa}z_1]\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^{M}\right)}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}e^{i\pi\alpha/\beta}}{e^{i\pi\gamma/\delta}[\tilde{\omega}^{\kappa}z_1]^{D\gamma-1}D\gamma} \tag{72}$$

$$=\frac{1}{e^{i\pi\gamma/\delta}[\tilde{\omega}^{\kappa}z_1]^{D\gamma}\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^{M}\right)}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}e^{i\pi\alpha/\beta}}{D\gamma} \tag{73}$$

$$=\frac{1}{[-z]\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^{M}\right)}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}e^{i\pi\alpha/\beta}}{D\gamma}. \tag{74}$$

where (71) follows from (69) and (72) follows from (70), while (73) follows from a consolidation of the factors $[\tilde{\omega}^{\kappa}z_1]$ and (74) follows from the facts that $\tilde{\omega}$ is a $[D\gamma]^{\text{th}}$ root of unity and $z_1^{D\gamma}=-e^{-\pi i\gamma/\delta}z$. Of course, we must have that $1/(\nu\theta(Q;\nu^M))$ is analytic at $\nu=\nu_\kappa=\tilde{\omega}^{\kappa}z_1$, and hence, $[\tilde{\omega}^{\kappa}z_1]^{M}\neq -Q^{k}$ for any $k\in\mathbb{Z}$, which is seen to be equivalent to $z\neq -[\omega^{j+1}Q^{k/M}]^{D\gamma}$ for any $j=0,\cdots,M-1$, and any $k\in\mathbb{Z}$. Furthermore, we must have $[\tilde{\omega}^{\kappa}z_1]\neq 0$, which is equivalent to $z\neq 0$. Thus, we require that $z\in\mathbb{C}^{*}\backslash\tilde{S}$, as hypothesized. The proposition is now proven.

Lemma 10. *Let $M=\operatorname{lcm}\{\beta,\delta\}$, and let $N\in\mathbb{N}$. Set $\Gamma_N=C_N-c_N=\partial A_N$ be the positively oriented boundary of the annular region A_N in $\mathbb{C}$ enclosed by the circular paths $C_N=((Q^{(N+1)/M}+Q^{N/M})/2)e^{i\phi}$ and $c_N=((Q^{(-N-1)/M}+Q^{-N/M})/2)e^{i\phi}$, where ϕ increases from 0 to 2π. Let $z\in\mathbb{C}^{*}\backslash\tilde{S}$, where $\tilde{S}\equiv\{-[\omega^{j+1}Q^{k/M}]^{D\gamma}\mid k\in\mathbb{Z},0\le j\le M-1\}$ and $\omega=e^{2\pi i/M}$.*

Let $\nu=\tilde{\omega}^{\kappa}z_1$ for $0\le\kappa\le D\gamma-1$ denote the roots of $z+\nu^{D\gamma}e^{i\pi\alpha/\beta}=0$, where $\tilde{\omega}=e^{2\pi i/[D\gamma]}$, z_0 is a $[D\gamma]^{th}$ root of z, and $z_1=e^{i\pi/[D\gamma]}e^{-i\pi\gamma/[\delta D\gamma]}z_0$. Choose N sufficiently large so that for $0\le\kappa\le D\gamma-1$ one has $\tilde{\omega}^{\kappa}z_1\in Int(A_N)$, the interior of A_N. Then, for $f_2(\nu)$ as in (62), one has

$$\begin{aligned}\int_{\Gamma_N}f_2(\nu)d\nu&=\int_{\Gamma_N}\frac{1}{\nu\theta(Q;\nu^M)}\frac{\nu^{B\alpha}e^{i\pi\alpha/\beta}}{\left(z+\nu^{D\gamma}e^{i\pi\gamma/\delta}\right)}d\nu\\&=2\pi i\frac{1}{\mu_Q^3}\frac{1}{M}\sum_{\ell=0}^{M-1}\sum_{k=-N}^{N}\left[\frac{(-1)^{k+1}}{Q^{k(k+1)/2}}\frac{[\omega^{\ell+1}Q^{k/M}]^{B\alpha}}{z+[\omega^{\ell+1}Q^{k/M}]^{D\gamma}}\right]\end{aligned} \tag{75}$$

$$+2\pi i\frac{e^{i\pi\alpha/\beta}}{(-z)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^{M}\right)}. \tag{76}$$

Proof. The integral over Γ_N yields $2\pi i$ times the enclosed residues, which occur at $\nu=\nu_\kappa=\tilde{\omega}^{\kappa}z_1$ for $0\le\kappa\le D\gamma-1$ in Int (A_N) the interior of A_N, as well as at the zeroes of $\theta(Q;\nu^M)$ in A_N, which by construction of Γ_N are $\nu=\nu_{k,\ell}=\omega^{\ell}e^{i\pi/M}Q^{k/M}$ for $-N\le k\le N$ and $0\le\ell\le M-1$. So the residue theorem gives

$$\begin{aligned}&\int_{\Gamma_N}\frac{1}{\nu\theta(Q;\nu^M)}\frac{\nu^{B\alpha}e^{i\pi\alpha/\beta}}{\left(z+\nu^{D\gamma}e^{i\pi\gamma/\delta}\right)}d\nu\\&=2\pi i\left(\sum_{\ell=0}^{M-1}\sum_{k=-N}^{N}\operatorname{Res}(f_2,\nu_{k,\ell})\right)+2\pi i\left(\sum_{\kappa=0}^{D\gamma-1}\operatorname{Res}(f_2,\nu_\kappa)\right)\end{aligned} \tag{77}$$

$$=2\pi i\frac{1}{\mu_Q^3}\frac{1}{M}\sum_{\ell=0}^{M-1}\sum_{k=-N}^{N}\left[\frac{(-1)^{k+1}}{Q^{k(k+1)/2}}\frac{[\omega^{\ell+1}Q^{k/M}]^{B\alpha}}{z+[\omega^{\ell+1}Q^{k/M}]^{D\gamma}}\right] \tag{78}$$

$$+2\pi i\frac{e^{i\pi\alpha/\beta}}{(-z)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^{M}\right)}, \tag{79}$$

where $\operatorname{Res}(f_2,\nu_{k,\ell})$ in (77) has been replaced by (62) to obtain the summation in (78) and where $\operatorname{Res}(f_2,\nu_\kappa)$ in (77) has been replaced by (65) to obtain the summation in (79). The expressions in (78) and (79) now give (75) and (76), and the lemma is proven. □

Lemma 11. *Let the assumptions in the first paragraph of Lemma 10 hold. Then,*

$$\lim_{N\longrightarrow\infty}\int_{C_N}\left|\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}\right||dv|=0, \quad (80)$$

$$\lim_{N\longrightarrow\infty}\int_{c_N}\left|\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}\right||dv|=0, \quad (81)$$

and hence,

$$\lim_{N\longrightarrow\infty}\int_{\Gamma_N}\left|\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}\right||dv|=0, \quad (82)$$

$$\lim_{N\longrightarrow\infty}\int_{\Gamma_N}\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}dv=0. \quad (83)$$

Proof. We show (80) and (81), which implies immediately that (82) and (83) hold. Now, the vanishing of each of the limits in (80) and (81) holds because $\theta(Q;v^M)$ grows sufficiently rapidly as N approaches infinity for $v\in C_N$ or $v\in c_N$. This rapid growth will follow directly from the identity (12). Designate $C_0=\{w \mid w=((Q^{1/M}+1)/2)e^{i\phi}, \phi\in[0,2\pi]\}$ as a reference circle of radius $\rho:=(Q^{1/M}+1)/2>1$ which by construction satisfies that $\theta(Q;w^M)$ never vanishes. This nonvanishing is the result of the fact that $1<\rho=|w|<Q^{1/M}$, whereby $1<|w^M|<Q$, along with the fact that all zeros of $\theta(Q;v^M)$ occur for $v^M=-Q^k$ for some $k\in\mathbb{Z}$, from (13). By continuity of $\theta(Q;w^M)$ on the compact set C_0, there exist constants b and B, each depending on Q, such that for all $w\in C_0$

$$0<b\le\left|\theta(Q;w^M)\right|\le B<\infty. \quad (84)$$

□

Note that $v\in C_N$ implies $\exists\ w\in C_0$ with $v=Q^{N/M}w$, and by (12), one has

$$\begin{aligned}\theta(Q;v^M)&=\theta\left(Q;\left[Q^{N/M}w\right]^M\right)=\theta(Q;Q^Nw^M)\\&=Q^{N(N+1)/2}w^{NM}\theta(Q;w^M),\end{aligned} \quad (85)$$

with $|\theta(Q;v^M)|=Q^{N(N+1)/2}\rho^{NM}|\theta(Q;w^M)|\ge Q^{N(N+1)/2}\rho^{NM} b$. Then for N_0 such that $([Q^{N_0/M}\rho]^{D\gamma}-|z|)>0$ and all $N\ge N_0$, one has

$$\begin{aligned}&\int_{C_N}\left|\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}\right||dv|\\&\quad=\int_{C_N}\frac{1}{|\theta(Q;v^M)|}\frac{|v|^{B\alpha}}{|z+v^{D\gamma}e^{i\pi\gamma/\delta}|}\frac{|dv|}{|v|}\\&\quad\le\frac{1}{(Q^{N(N+1)/2}\rho^{NM}b)}\frac{\left[Q^{N/M}\rho\right]^{B\alpha}}{\left(\left[Q^{N/M}\rho\right]^{D\gamma}-|z|\right)}2\pi,\end{aligned} \quad (86)$$

which approaches 0 as N approaches infinity, as $Q^{N(N+1)/2}$ is the dominant term. Similarly, $v\in c_N$ implies $\exists\ w\in C_0$ with $v=Q^{(-N-1)/M}w$, and by (12), one has

$$\begin{aligned}\theta(Q;v^M)&=\theta\left(Q;\left[Q^{(-N-1)/M}w\right]^M\right)\\&=\theta\left(Q;Q^{(-N-1)}w^M\right)\\&=Q^{(-N-1)(-N)/2}w^{M(-N-1)}\theta(Q;w^M),\end{aligned} \quad (87)$$

with $|\theta(Q;v^M)|=Q^{N(N+1)/2}\rho^{M(-N-1)}|\theta(Q;w^M)|\ge Q^{N(N+1)/2}\rho^{M(-N-1)}b$.

Thus, for N_1 such that $(|z|-[Q^{(-N_1-1)/M}\rho]^{D\gamma})>0$ and all $N\ge N_1$, one has

$$\begin{aligned}&\int_{c_N}\left|\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}\right||dv|\\&\quad=\int_{c_N}\frac{1}{|\theta(Q;v^M)|}\frac{|v|^{B\alpha}}{|z+v^{D\gamma}e^{i\pi\gamma/\delta}|}\frac{|dv|}{|v|}\\&\quad\le\frac{1}{(Q^{N(N+1)/2}\rho^{M(-N-1)}b)}\frac{\left[Q^{(-N-1)/M}\rho\right]^{B\alpha}}{\left(|z|-\left[Q^{(-N-1)/M}\rho\right]^{D\gamma}\right)}2\pi,\end{aligned} \quad (88)$$

which also vanishes as N approaches infinity, as $Q^{N(N+1)/2}$ is the dominant term. The lemma is now shown.

With Lemma 11 in mind, we record the following corollary which will be utilized later.

Corollary 12. *Let $S\equiv S(\phi_1,\phi_2)=\{v\in\mathbb{C}^* \mid \phi_1\le\arg(v)\le\phi_2\}\cup\{0\}$ be the sector in $\mathbb{C}$ emanating from the origin with argument falling in the interval $[\phi_1,\phi_2]$ with $\phi_2-\phi_1\le 2\pi$. Let $s_N\equiv S\cap c_N$ and $S_N\equiv S\cap C_N$, where z, C_N, and c_N are as in Lemma 10. Then*

$$\lim_{N\longrightarrow\infty}\int_{S_N}\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}dv=0, \quad (89)$$

$$\lim_{N\longrightarrow\infty}\int_{s_N}\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}dv=0. \quad (90)$$

Proof. One has

$$0\le\left|\int_{S_N}\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}dv\right| \quad (91)$$

$$\le\int_{S_N}\left|\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}\right||dv| \quad (92)$$

$$\le\int_{C_N}\left|\frac{1}{v\theta(Q;v^M)}\frac{v^{B\alpha}e^{i\pi\alpha/\beta}}{(z+v^{D\gamma}e^{i\pi\gamma/\delta})}\right||dv|. \quad (93)$$

□

Since the limit as N approaches infinity of (93) vanishes, by (80), the vanishing in (89) holds. The vanishing in (90) holds by replacing S_N with s_N and C_N with c_N in (91)–(93), taking limits as N approaches infinity, and relying on (81). This proves the corollary.

We have now arrived at a preliminary version of the main theorem of this study. It is "preliminary" in that in certain cases each side of the main equality (98) and (99) in Theorem 13 immediately below reduces to an identically 0 function, and it will be necessary later to give nonidentically vanishing criteria in our main result Theorem 23, where we will also relate the expression in (99) to the Fourier transform of a function naturally generated by $f_{\mu,\lambda}(t)$. First, we pause to catalogue the current set of results.

Theorem 13. *For $q>1$, let $f_{\mu,\lambda}(t)$ be as in (1) and $\tilde{f}_{\mu,\lambda}(t)$ be as in (14), respectively, where we assume $\mu\in\mathbb{Q}$ and $\lambda\in\mathbb{Q}^+$. Let $z\in\mathbb{C}^*\backslash\tilde{S}$ where $\tilde{S}\equiv\{-[\omega^{j+1}Q^{k/M}]^{D\gamma}\mid k\in\mathbb{Z}, 0\le j\le M-1\}$. Then,*

$$\frac{1}{\mu_Q^3}\frac{1}{M}\sum_{\ell=0}^{M-1}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[\omega^{\ell+1}Q^{k/M}\right]^{B\alpha}}{z+\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}}\right] \tag{94}$$

$$=\frac{e^{i\pi\alpha/\beta}}{(-z)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^M\right)}, \tag{95}$$

where μ_Q is given by (11), where $\theta(Q;z)$ is the Jacobi theta function, where

$$\begin{aligned}&\frac{\alpha}{\beta}=\frac{\mu+1}{2} && \frac{\gamma}{\delta}=\frac{\lambda}{2} && M=\operatorname{lcm}\{\beta,\delta\}\\ &Q=q^{2/\lambda}=q^{\delta/\gamma} && B=\frac{M}{\beta} && D=\frac{M}{\delta}\\ &\omega=e^{2\pi i/M} && && \tilde{\omega}=e^{2\pi i/[D\gamma]}\end{aligned}, \tag{96}$$

and where $\alpha\in\mathbb{Z}$ and $\beta,\gamma,\delta\in\mathbb{N}$ with α/β and γ/δ in reduced form. M is taken to be the least common multiple of β and δ. Also, for z_0 any fixed $[D\gamma]^{th}$ root of z, one has that z_1 in (95) is given by

$$z_1=e^{i\pi/[D\gamma]}e^{-i\pi\gamma/[\delta D\gamma]}z_0,\quad \text{whereby}\quad z_1^{D\gamma}=-e^{-\pi i\gamma/\delta}z. \tag{97}$$

Setting $z=ix$ (94) for $x\in\mathbb{R}$ and requiring that $\delta\not\equiv 0 \bmod 4$ give the following relation of the weighted average of the rotations of the Fourier transforms $\mathcal{F}[\tilde{f}_{\mu,\lambda}(t)](x)$ with the average of the rotations of $z_3/\theta(Q;z_3^M)$:

$$\frac{1}{M}\left[\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](x)+\sum_{\ell=0}^{M-2}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{[\omega^{\ell+1}]^{D\gamma}}\right)\right] \tag{98}$$

$$=\frac{\mu_Q^3e^{i\pi\alpha/\beta}}{\sqrt{2\pi}}\frac{1}{(-ix)}\left(\frac{1}{D\gamma}\left[\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^M\right)}\right]\right). \tag{99}$$

Here, $\mathcal{F}[\tilde{f}_{\mu,\lambda}(t)](x/[\omega^{\ell+1}]^{D\gamma})$ is given by (17). Also, for z_2 any fixed $[D\gamma]^{th}$ root of ix, one has that z_3 in (99) is given by

$$z_3=e^{i\pi/[D\gamma]}e^{-i\pi\gamma/[\delta D\gamma]}z_2,\ \text{whereby}\ z_3^{D\gamma}=-e^{-\pi i\gamma/\delta}ix. \tag{100}$$

Proof. Integrating the integrand (64) over the oriented boundary Γ_N of the annular region A_N as in Lemma 10 gives expressions (75) and (76). Taking the limit of (75) and (76) as N approaches infinity, and relying on Lemma 11 gives

$$0=2\pi i\frac{1}{\mu_Q^3}\frac{1}{M}\sum_{\ell=0}^{M-1}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^{k+1}}{Q^{k(k+1)/2}}\frac{\left[\omega^{\ell+1}Q^{k/M}\right]^{B\alpha}}{z+\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}}\right] \tag{101}$$

$$+2\pi i\frac{e^{i\pi\alpha/\beta}}{(-z)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^M\right)}, \tag{102}$$

where z_1 is given by (97). Dividing (101) and (102) by $2\pi i$ and moving the double summation to the left side of the equality give (94) and (95). Now, multiplying (94) and (95) by $\mu_Q^3/\sqrt{2\pi}$ gives

$$\frac{1}{M}\sum_{\ell=0}^{M-1}\left[\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[\omega^{\ell+1}Q^{k/M}\right]^{B\alpha}}{z+\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}}\right]\right] \tag{103}$$

$$=\frac{\mu_Q^3}{\sqrt{2\pi}}\frac{e^{i\pi\alpha/\beta}}{(-z)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_1]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_1]^M\right)}. \tag{104}$$

□

Setting $z=ix$ in (103) and (104) gives that (97) becomes (100), and we then also replace z_1 in (104) by z_3 to obtain (105)–(107) below. Factoring out the powers of $\omega^{\ell+1}$ from (103) now yields

$$\frac{1}{M}\sum_{\ell=0}^{M-1}\left[\frac{\left[\omega^{\ell+1}\right]^{B\alpha}}{1}\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k/M}\right]^{B\alpha}}{ix+\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}}\right]\right] \tag{105}$$

$$=\frac{1}{M}\sum_{\ell=0}^{M-1}\left[\frac{\left[\omega^{\ell+1}\right]^{B\alpha}}{\left[\omega^{\ell+1}\right]^{D\gamma}}\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k/M}\right]^{B\alpha}}{ix/[\omega^{\ell+1}]^{D\gamma}+\left[Q^{k/M}\right]^{D\gamma}}\right]\right] \tag{106}$$

$$=\frac{\mu_Q^3}{\sqrt{2\pi}}\frac{e^{i\pi\alpha/\beta}}{(-ix)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^M\right)}. \tag{107}$$

One now recognizes that in (106)

$$\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k/M}\right]^{B\alpha}}{ix/[\omega^{\ell+1}]^{D\gamma}+\left[Q^{k/M}\right]^{D\gamma}}\right]$$
$$=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k/M}\right]^{(M/\beta)\alpha}}{ix/[\omega^{\ell+1}]^{D\gamma}+\left[Q^{k/M}\right]^{(M/\delta)\gamma}}\right]$$
$$=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k}\right]^{(\alpha/\beta)}}{ix/[\omega^{\ell+1}]^{D\gamma}+\left[Q^{k}\right]^{(\gamma/\delta)}}\right] \tag{108}$$

$$=\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(x/\left[\omega^{\ell+1}\right]^{D\gamma}\right), \tag{109}$$

where (109) follows from (17).

Relying on (109) one sees that (106)-(107) becomes

$$\frac{1}{M}\sum_{\ell=0}^{M-1}\left[\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(x/\left[\omega^{\ell+1}\right]^{D\gamma}\right)\right] \tag{110}$$

$$=\frac{\mu_Q^3}{\sqrt{2\pi}}\frac{e^{i\pi\alpha/\beta}}{(-ix)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^{M}\right)}, \tag{111}$$

which is equivalent to (98) and (99) when one observes that for $\ell=M-1$ in (110) one has $\omega^{\ell+1}=\omega^M=1$.

At this point, we have shown (98) and (99) only for $x\neq 0$, so we next handle the case that $x=0$. Notice that (108) and (109) are defined at $x=0$ via

$$\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(x/\left[\omega^{\ell+1}\right]^{D\gamma}\right)\Big|_{x=0}=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k}\right]^{(\alpha/\beta)}}{0+\left[Q^{k}\right]^{(\gamma/\delta)}}\right] \tag{112}$$

$$=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{\left(-Q^{\alpha/\beta-\gamma/\delta-1}\right)^k}{Q^{k(k-1)/2}}\right] \tag{113}$$

$$=\frac{1}{\sqrt{2\pi}}\theta\left(Q;-Q^{\alpha/\beta-\gamma/\delta-1}\right), \tag{114}$$

where (114) follows from (10). Now, at $x=0$, from (114) one observes that (110) and (111) still hold in the sense that it becomes

$$\frac{1}{M}\left[\sum_{\ell=0}^{M-1}\left[\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\right]\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{[\omega^{\ell+1}]^{D\gamma}}\right)\right]\Bigg|_{x=0}=\frac{1}{M}\left[\sum_{\ell=0}^{M-1}\left[\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\right]\right]\left(\frac{1}{\sqrt{2\pi}}\theta\left(Q;-Q^{\alpha/\beta-\gamma/\delta-1}\right)\right) \tag{115}$$

$$=0=\lim_{x\longrightarrow 0}\left[\frac{\mu_Q^3}{\sqrt{2\pi}}\frac{e^{i\pi\alpha/\beta}}{(-ix)}\frac{1}{D\gamma}\sum_{\kappa=0}^{D\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^{M}\right)}\right], \tag{116}$$

where the vanishing of (115) follows from the vanishing of (159) which is proven in the paragraph containing (164) in Proposition 20 below (with n set to 0) and the vanishing of the limit in (116) follows from Corollary 32 below. Finally the requirement that $\delta\neq 0$ mod 4 is equivalent to the condition that $z=ix$ does not belong to $\tilde{S}$ for any $x\in\mathbb{R}$ by Lemma 14 below. The theorem is now proven.

The following lemma gives a simple characterization for having some x with $ix\in\tilde{S}$ (which is to be avoided in Theorem 13).

Lemma 14. *In the setting of Theorem 13, with the notation as in (96), one has*

$$\exists\, x\in\mathbb{R}\text{ such that } ix\in\tilde{S}=\left\{-\left[\omega^{j+1}Q^{k/M}\right]^{D\gamma}\mid j,k\in\mathbb{Z}\right\} \tag{117}$$

$$\Longleftrightarrow \delta=0 \bmod 4. \tag{118}$$

Proof. Observe that the existence of an $x\in\mathbb{R}$ with $ix\in\tilde{S}$ is equivalent to the existence of a $j\in\mathbb{Z}$ with either $i=[\omega^{j+1}]^{D\gamma}$ (in which case $x=-[Q^{k/M}]^{D\gamma}$ for some $k\in\mathbb{Z}$) or $i=-[\omega^{j+1}]^{D\gamma}$ (in which case $x=[Q^{k/M}]^{D\gamma}$ for some $k\in\mathbb{Z}$).

Since $\omega=\exp(2\pi i/M)$, one has

$$\begin{aligned}&\exists\, x\in\mathbb{R}\text{ with } ix\in\tilde{S}\\ &\Longleftrightarrow \exists\, j\in\mathbb{Z}\text{ with }\pm i=\left[\omega^{j+1}\right]^{D\gamma}\\ &\Longleftrightarrow \exists\, j\in\mathbb{Z},\, m\in\{1,3\},\text{ and}\\ &\quad n\in\mathbb{Z}\text{ with } e^{\pi i m/2}e^{2\pi i n}=e^{2\pi i(j+1)D\gamma/M}\end{aligned} \tag{119}$$

$$\Longleftrightarrow \exists\, j\in\mathbb{Z},\, m\in\{1,3\},\text{ and } n\in\mathbb{Z}\text{ with }\frac{m}{2}+2n=2(j+1)\frac{\gamma}{\delta} \tag{120}$$

$$\Longleftrightarrow \exists\, j\in\mathbb{Z},\, m\in\{1,3\},\text{ and } n\in\mathbb{Z}\text{ with }\delta[m+4n]=4(j+1)\gamma, \tag{121}$$

where (120) follows from the fact that $D=M/\delta$. Now from (121), one has that δ must be divisible by 4. Thus,

$$\exists\, x\in\mathbb{R}\text{ with } ix\in\tilde{S}\Rightarrow\delta=0 \bmod 4. \tag{122}$$

Conversely, if $\delta=0$ mod 4, then $\exists\, p\in\mathbb{N}$ with $\delta=4p$. Since γ and δ have no common factors, one concludes that γ is not divisible by 2 or 4. Thus, $\gamma=[m+4n]$ for some $m\in\{1,3\}$ and some $n\in\mathbb{Z}$. Hence,

$$4p\gamma=\delta\gamma=\delta[m+4n]. \tag{123}$$

Setting $j=p-1$ gives that (121) holds. We conclude

$$\delta = 0 \mod 4 \Rightarrow \exists\, x \in \mathbb{R} \text{ with } ix \in \tilde{S}. \tag{124}$$

Equations (122) and (124) now give the equivalence (117) and (118), and the lemma is proven. □

3. The Functions Naturally Generated by the $f_{\mu,\lambda}(t)$

In this section, we again assume the notation of the previous section. In particular, the notation in (96) holds. Namely, $\mu, \lambda \in \mathbb{Q}$ are rational with $\lambda > 0$, with $x, t \in \mathbb{R}$. Also $(\mu+1)/2 = \alpha/\beta$ and $\lambda/2 = \gamma/\delta$ with $\alpha \in \mathbb{Z}$ and $\beta, \gamma, \delta \in \mathbb{N}$, where both α/β and γ/δ are in a reduced form. Finally, $M = \operatorname{lcm}\{\beta, \delta\}$ is the least common multiple of β and δ, $B = M/\beta$, $D = M/\delta$, and $\omega = e^{2\pi i/M}$. Let $Q = q^{2/\lambda}$, and let $0 \le \ell \le M-1$. With this notation in mind, we are able to make the following definitions.

Definition 15. Assume that $\delta \ne 0 \mod 4$. Then by Lemma 14, one sees from (117), (118), and (119) that $\nexists\, j \in \mathbb{Z}$ *with* $\pm i = [\omega^{j+1}]^{D\gamma}$. Hence, the real part $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) \ne 0$. For the real part $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) > 0$ define

$$\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \equiv \begin{cases} (+1)\displaystyle\sum_{k\in\mathbb{Z}} \frac{(-1)^k \exp\left(-\left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right)}{q^{k(k-\mu)/\lambda}} = (+1) f_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right), & \text{for } t \ge 0, \\ 0, & \text{for } t < 0, \end{cases} \tag{125}$$

and for real part $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) < 0$ define

$$\tilde{f}^{\le}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \equiv \begin{cases} 0, & \text{for } t > 0, \\ (-1)\displaystyle\sum_{k\in\mathbb{Z}} \frac{(-1)^k \exp\left(-\left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right)}{q^{k(k-\mu)/\lambda}} = (-1) f_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right), & \text{for } t \le 0, \end{cases} \tag{126}$$

while for $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) < 0$ we also define

$$\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \equiv \chi_{(-\infty,0)}(t)\, \tilde{f}^{\le}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \tag{127}$$

$$= \begin{cases} 0, & \text{for } t \ge 0, \\ (-1) f_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right), & \text{for } t < 0, \end{cases} \tag{128}$$

where $\chi_{(-\infty,0)}(t)$ is the characteristic function of the interval $(-\infty, 0)$. We emphasize the $(+1)$ coefficient in (125) versus the (-1) coefficient in (126) and (128).

Note that for $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) < 0$

$$\lim_{t\to 0^-} \tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) = \tilde{f}^{\le}_{\mu,\lambda}(0) = -f_{\mu,\lambda}(0), \tag{129}$$

and for $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) > 0$

$$\lim_{t\to 0^+} \tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) = \tilde{f}_{\mu,\lambda}(0) = f_{\mu,\lambda}(0). \tag{130}$$

Definition 16. Assume $\delta \ne 0 \mod 4$. The function naturally generated by $f_{\mu,\lambda}(t)$ is given by

$$\mathscr{W}_{\mu,\lambda}(t) \equiv \sum_{\ell=0}^{M-1} \left[\omega^{\ell+1}\right]^{B\alpha} \tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \tag{131}$$

$$= \sum_{\{-\}} \left[\omega^{\ell+1}\right]^{B\alpha} \tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) + \sum_{\{+\}} \left[\omega^{\ell+1}\right]^{B\alpha} \tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right), \tag{132}$$

where the leftmost summation over the negative index $\{-\}$ in (132) stands for summation over the indices ℓ with $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) < 0$ and the rightmost summation over the positive index $\{+\}$ in (125) stands for summation over the indices ℓ with $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) > 0$.

While the $\mathscr{W}_{\mu,\lambda}(t)$ are constructed here in Definition 16, they were originally obtained as the inverse Fourier transform of the expression in (98) in Theorem 13 (scaled by M). In this sense, the $\mathscr{W}_{\mu,\lambda}(t)$ are natural. We point out that the $\mathscr{W}_{\mu,\lambda}(t)$ may vanish identically, as discussed in Proposition 21 below. However, there are a wealth of

$\mathscr{W}_{\mu,\lambda}(t)$ that are not vanishing identically, as characterized in Proposition 21. In any case, the $\mathscr{W}_{\mu,\lambda}(t)$ have useful properties that are now recorded as a series of propositions. The first observation is that $\mathscr{W}_{\mu,\lambda}(t)$ is real valued.

Proposition 17. *For all $t \in \mathbb{R}$, $\mathscr{W}_{\mu,\lambda}(t) \in \mathbb{R}$.*

Proof. Observe that for $[\omega^{\ell+1}]^{D\gamma} = \mathscr{R}([\omega^{\ell+1}]^{D\gamma}) + i\mathscr{I}([\omega^{\ell+1}]^{D\gamma})$ one has $\text{Conj}([\omega^{\ell+1}]^{D\gamma}) = \mathscr{R}([\omega^{\ell+1}]^{D\gamma}) - i\mathscr{I}([\omega^{\ell+1}]^{D\gamma})$, and hence, conjugation preserves the sign of $\mathscr{R}([\omega^{\ell+1}]^{D\gamma})$. Thus, conjugation also preserves the sets $\{-\} \equiv \{\ell \mid \mathscr{R}([\omega^{\ell+1}]^{D\gamma}) < 0\}$ and $\{+\} \equiv \{\ell \mid \mathscr{R}([\omega^{\ell+1}]^{D\gamma}) > 0\}$. This allows us to conclude that

$$\text{Conj}\left[\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right)\right] = \tilde{f}_{\mu,\lambda}\left(\left[\bar{\omega}^{\ell+1}\right]^{D\gamma} t\right) \tag{133}$$

independently of the (-1) factor in (128) versus the $(+1)$ factor in (125), where $\bar{\omega} = \text{Conj}(\omega) = e^{-2\pi i/M}$. Hence, from (132), one has

$$\begin{aligned}\text{Conj}\left(\mathscr{W}_{\mu,\lambda}(t)\right) &= \sum_{\{-\}}\left[\bar{\omega}^{\ell+1}\right]^{B\alpha}\tilde{f}_{\mu,\lambda}\left(\left[\bar{\omega}^{\ell+1}\right]^{D\gamma} t\right) \\ &\quad + \sum_{\{+\}}\left[\bar{\omega}^{\ell+1}\right]^{B\alpha}\tilde{f}_{\mu,\lambda}\left(\left[\bar{\omega}^{\ell+1}\right]^{D\gamma} t\right) \\ &= \sum_{\ell=0}^{M-1}\left[\bar{\omega}^{\ell+1}\right]^{B\alpha}\tilde{f}_{\mu,\lambda}\left(\left[\bar{\omega}^{\ell+1}\right]^{D\gamma} t\right) \\ &= \sum_{\ell=0}^{M-1}\left[\omega^{\ell+1}\right]^{B\alpha}\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right),\end{aligned} \tag{134}$$

where the last equation in (134) follows since conjugation permutes each of the sets $\{\omega^\ell \mid 0 \le \ell \le M-1\}$, $\{\omega^\ell \mid \ell \in \{-\}\}$, and $\{\omega^\ell \mid \ell \in \{+\}\}$. The proposition now follows. □

The second result gives smoothness.

Proposition 18. *$\mathscr{W}_{\mu,\lambda}(t)$ is $\mathscr{C}^\infty$ at $t = 0$, and hence, $\mathscr{W}_{\mu,\lambda}(t)$ is in $\mathscr{C}^\infty(\mathbb{R})$. Furthermore, $\mathscr{W}_{\mu,\lambda}(t)$ is Schwartz.*

Proof. Away from $t = 0$, $\mathscr{W}_{\mu,\lambda}(t)$ is $\mathscr{C}^\infty$ via (125)-(126). From (132), note that for $t < 0$, one has $\mathscr{W}_{\mu,\lambda}(t) = \sum_{\{-\}}[\omega^{\ell+1}]^{B\alpha}\tilde{f}_{\mu,\lambda}([\omega^{\ell+1}]^{D\gamma} t)$ and for $t \ge 0$ one has $\mathscr{W}_{\mu,\lambda}(t) = \sum_{\{+\}}[\omega^{\ell+1}]^{B\alpha}\tilde{f}_{\mu,\lambda}([\omega^{\ell+1}]^{D\gamma} t)$. Smoothness will follow at $t = 0$, by showing that $\lim_{t\to 0^-}\mathscr{W}^{(n)}_{\mu,\lambda}(t) = \lim_{t\to 0^+}\mathscr{W}^{(n)}_{\mu,\lambda}(t)$ for all $n \in \mathbb{N}_0$. From (125) and (126), one has

$$\begin{aligned}\lim_{t\to 0^-}\mathscr{W}^{(n)}_{\mu,\lambda}(t) &= \lim_{t\to 0^-}\sum_{\{-\}}\left[\omega^{\ell+1}\right]^{B\alpha}\frac{d^n}{dt^n}\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \\ &= \lim_{t\to 0^-}\sum_{\{-\}}\left[\omega^{\ell+1}\right]^{B\alpha}(-1)^n\left[\omega^{\ell+1}\right]^{D\gamma n} \\ &\quad\cdot \tilde{f}_{\mu+n\lambda,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \\ &= \sum_{\{-\}}\left[\omega^{\ell+1}\right]^{B\alpha}(-1)^n\left[\omega^{\ell+1}\right]^{D\gamma n}\tilde{f}^{\le}_{\mu+n\lambda,\lambda}(0)\end{aligned} \tag{135}$$

$$= \sum_{\{-\}}\left[\omega^{\ell+1}\right]^{B\alpha}(-1)^n\left[\omega^{\ell+1}\right]^{D\gamma n}(-1)f_{\mu+n\lambda,\lambda}(0), \tag{136}$$

where (135) follows directly from differentiating (126) and (136) also follows from (126). Also,

$$\begin{aligned}\lim_{t\to 0^+}\mathscr{W}^{(n)}_{\mu,\lambda}(t) &= \lim_{t\to 0^+}\sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha}\frac{d^n}{dt^n}\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \\ &= \lim_{t\to 0^+}\sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha}(-1)^n\left[\omega^{\ell+1}\right]^{D\gamma n} \\ &\quad\cdot \tilde{f}_{\mu+n\lambda,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \\ &= \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha}(-1)^n\left[\omega^{\ell+1}\right]^{D\gamma n}\tilde{f}_{\mu+n\lambda,\lambda}(0)\end{aligned} \tag{137}$$

$$= \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha}(-1)^n\left[\omega^{\ell+1}\right]^{D\gamma n}f_{\mu+n\lambda,\lambda}(0), \tag{138}$$

where (135) follows directly from differentiating (125) and (138) follows from (1). Equality of (136) with (138) would be equivalent with

$$0 = \left[\sum_{\{-\}}\left[\omega^{\ell+1}\right]^{B\alpha}\left[\omega^{\ell+1}\right]^{D\gamma n} + \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha}\left[\omega^{\ell+1}\right]^{D\gamma n}\right]f_{\mu+n\lambda,\lambda}(0). \tag{139}$$

□

Now if $B\alpha + D\gamma n \ne 0 \mod M$, then

$$\sum_{\{-\}}\left[\omega^{\ell+1}\right]^{B\alpha}\left[\omega^{\ell+1}\right]^{D\gamma n} + \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha}\left[\omega^{\ell+1}\right]^{D\gamma n} = 0 \tag{140}$$

from Lemma 6. Hence (139) holds. On the other hand, if $B\alpha + D\gamma n = 0 \mod M$, then

$$\sum_{\{-\}}\left[\omega^{\ell+1}\right]^{B\alpha}\left[\omega^{\ell+1}\right]^{D\gamma n} + \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha}\left[\omega^{\ell+1}\right]^{D\gamma n} = M \ne 0 \tag{141}$$

from Lemma 6. In this case, one has that for some $k \in \mathbb{Z}$

$$Mk = B\alpha + D\gamma n = M\left[\frac{\alpha}{\beta} + \frac{\gamma}{\delta}n\right] \Leftrightarrow k = \left[\frac{\alpha}{\beta} + \frac{\gamma}{\delta}n\right] \in \mathbb{Z} \tag{142}$$

$$\Leftrightarrow k - 1 = \left[\frac{\alpha}{\beta} - 1 + \frac{\gamma}{\delta}n\right] \in \mathbb{Z}, \tag{143}$$

where we have used (96) to rewrite $B = M/\beta, D = M/\delta$ in the second equality of (142). Now from Lemma 2.4 of [2], one has $f_{\mu+n\lambda}(0) = \theta(q^{2/\lambda}; -q^{(\mu+n\lambda-1)/\lambda}) = \theta(Q; -Q^{(\mu+n\lambda-1)/2})$, where from (96) we have used $q^{2/\lambda} = Q$. From (13), it follows that $\theta(Q; -Q^{(\mu+n\lambda-1)/2}) = 0$ if and only if there is a $k \in \mathbb{Z}$ with $k = (\mu + n\lambda - 1)/2$. Now,

$$k = \frac{\mu + n\lambda - 1}{2} \Leftrightarrow k = \frac{\mu + 1}{2} - 1 + n\frac{\lambda}{2} \Leftrightarrow k = \left[\frac{\alpha}{\beta} - 1 + \frac{\gamma}{\delta}n\right] \in \mathbb{Z}, \tag{144}$$

where (96) was used to rewrite $(\mu + 1)/2 = \alpha/\beta$ and $\lambda/2 = \gamma/\delta$. Observing equality of the rightmost expression in (144) with the rightmost expression (143), one has that in (139)

$$\left[\sum_{\ell=0}^{M-1} \left[\omega^{\ell+1}\right]^{B\alpha} \left[\omega^{\ell+1}\right]^{D\gamma n}\right] \neq 0 \text{ precisely when } f_{\mu+n\lambda,\lambda}(0) = 0. \tag{145}$$

Thus, in all cases, (139) holds, whence equality of (136) and (138) holds. We conclude that $\mathscr{W}_{\mu,\lambda}(t)$ is $\mathscr{C}^\infty$ at $t = 0$ and thus on $\mathbb{R}$.

Finally, the fact that $\mathscr{W}_{\mu,\lambda}(t)$ is Schwartz follows from (1) the fact that it is in $\mathscr{C}^\infty(\mathbb{R})$ and (2) from the fact that for $|t|$ sufficiently large one has $|\mathscr{W}_{\mu,\lambda}(t)| \leq K_1 |t|^{-K_2 \ln(|t|) + K_3}$ for constants $K_1, K_2 > 0$ and $K_3 \in \mathbb{R}$, which in turn follows from the expressions (125) and (126) and from Proposition 8.1 of [2]. The proposition is now proven.

One consequence of Proposition 18 is that $\mathscr{W}_{\mu,\lambda}(t)$ has a Fourier transform which is Schwartz. The following proposition allows us to observe that $\mathscr{W}_{\mu,\lambda}(t)$ was defined so that its Fourier transform would be given by (98) (up to the constant factor $1/M$).

Proposition 19. *The Fourier transform of $\mathscr{W}_{\mu,\lambda}(t)$ is given by*

$$\mathscr{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x) = \left[\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](x) + \sum_{\ell=0}^{M-2} \left[\omega^{\ell+1}\right]^{B\alpha - D\gamma} \cdot \mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{\left[\omega^{\ell+1}\right]^{D\gamma}}\right)\right]. \tag{146}$$

Proof. From (131), one has

$$\mathscr{F}\left[\mathscr{W}_{\mu,\lambda}\right](x) = \sum_{\ell=0}^{M-1} \left[\omega^{\ell+1}\right]^{B\alpha} \mathscr{F}\left[\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right)\right](x). \tag{147}$$

We next evaluate $\mathscr{F}[\tilde{f}_{\mu,\lambda}([\omega^{\ell+1}]^{D\gamma} t)](x)$ in (147). Observe that for $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) > 0$ one has

$$\begin{aligned}
&\mathscr{F}\left[\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right)\right](x) \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ixt} \sum_{k=-\infty}^{\infty} (-1)^k \frac{\exp\left(-\left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right)}{q^{k(k-\mu)/\lambda}} dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \int_0^\infty \exp\left(-ixt - \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right) dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{\exp\left(-ixt - \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right)}{-ix - \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}}\Bigg|_0^\infty \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{1}{ix + \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}}.
\end{aligned} \tag{148}$$

Similarly, for $\mathscr{R}([\omega^{\ell+1}]^{D\gamma}) < 0$, one has

$$\begin{aligned}
&\mathscr{F}\left[\tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right)\right](x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-ixt} (-1) \sum_{k=-\infty}^{\infty} (-1)^k \frac{\exp\left(-\left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right)}{q^{k(k-\mu)/\lambda}} dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \int_{-\infty}^0 (-1) \exp\left(-ixt - \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right) dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{(-1) \exp\left(-ixt - \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma} t\right)}{-ix - \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}}\Bigg|_{-\infty}^0 \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{1}{ix + \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}}.
\end{aligned} \tag{149}$$

From the matching forms of (148) and (149), one sees that (147) becomes

$$\mathscr{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x) = \sum_{\ell=0}^{M-1} \left[\omega^{\ell+1}\right]^{B\alpha} \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \cdot \frac{1}{ix + \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}}. \tag{150}$$

Noticing that

$$\frac{(-1)^k}{q^{k(k-\mu)/\lambda}} = \frac{(-1)^k q^{k(\mu+1)/\lambda}}{q^{k(k+1)/\lambda}} = \frac{(-1)^k\left[q^{2/\lambda}\right]^{k(\mu+1)/2}}{\left(q^{2/\lambda}\right)^{k(k+1)/2}} = \frac{(-1)^k\left[Q^k\right]^{(\mu+1)/2}}{Q^{k(k+1)/2}} = \frac{(-1)^k\left[Q^{k/M}\right]^{M(\alpha/\beta)}}{Q^{k(k+1)/2}} = \frac{(-1)^k\left[Q^{k/M}\right]^{B\alpha}}{Q^{k(k+1)/2}} \quad (151)$$

reexpresses (150) as

$$\mathcal{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x) = \sum_{\ell=0}^{M-1}\left[\omega^{\ell+1}\right]^{B\alpha}\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\frac{(-1)^k\left[Q^{k/M}\right]^{B\alpha}}{Q^{k(k+1)/2}} \cdot \frac{1}{ix+\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}} \quad (152)$$

$$= \sum_{\ell=0}^{M-1}\frac{\left[\omega^{\ell+1}\right]^{B\alpha}}{\left[\omega^{\ell+1}\right]^{D\gamma}}\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k/M}\right]^{B\alpha}}{ix/\left[\omega^{\ell+1}\right]^{D\gamma}+\left[Q^{k/M}\right]^{D\gamma}} = \sum_{\ell=0}^{M-1}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(x/\left[\omega^{\ell+1}\right]^{D\gamma}\right), \quad (153)$$

where (153) follows from (17). Now, (153) gives (146) and the proposition is proven. □

Another property of $\mathscr{W}_{\mu,\lambda}(t)$ is that all of its moments vanish.

Proposition 20. *All moments of $\mathscr{W}_{\mu,\lambda}(t)$ vanish. That is,*

$$\int_{-\infty}^{\infty} t^n \mathscr{W}_{\mu,\lambda}(t)dt = 0, \quad \forall\ n \in \mathbb{N}\cup\{0\}. \quad (154)$$

Equivalently, all the derivatives of $\mathcal{F}[\mathscr{W}_{\mu,\lambda}(t)](x)$ satisfy

$$\left.\frac{d^n}{dx^n}\mathcal{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x)\right|_{x=0} = 0. \quad (155)$$

Proof. We proceed by showing (155). Differentiating (152) yields

$$\frac{d^n}{dx^n}\left(\mathcal{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x)\right) = (-i)^n n!\sum_{\ell=0}^{M-1}\left[\frac{\left[\omega^{\ell+1}\right]^{B\alpha}}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k/M}\right]^{B\alpha}}{\left(ix+\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}\right)^{n+1}}\right]\right]. \quad (156)$$

Evaluating at $x=0$ gives

$$\left.\frac{d^n}{dx^n}\mathcal{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x)\right|_{x=0} = (-i)^n n!\sum_{\ell=0}^{M-1}\left[\frac{\left[\omega^{\ell+1}\right]^{B\alpha}}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^{k/M}\right]^{B\alpha}}{\left(\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}\right)^{n+1}}\right]\right] = \frac{(-i)^n n!}{\sqrt{2\pi}}\sum_{\ell=0}^{M-1}\left[\frac{\left[\omega^{\ell+1}\right]^{B\alpha}}{\left[\omega^{\ell+1}\right]^{D\gamma(n+1)}}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\left[Q^{k/M}\right]^{B\alpha-D\gamma(n+1)}\right]\right] \quad (157)$$

$$= \frac{(-i)^n n!}{\sqrt{2\pi}}\sum_{\ell=0}^{M-1}\left[\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma(n+1)}\theta\left(Q;-Q^{\alpha/\beta-(n+1)\gamma/\delta-1}\right)\right] \quad (158)$$

$$= \frac{(-i)^n n!}{\sqrt{2\pi}}\left[\sum_{\ell=0}^{M-1}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma(n+1)}\right]\theta\left(Q;-Q^{\alpha/\beta-(n+1)\gamma/\delta-1}\right). \quad (159)$$

Here, movement from (157) to (158) is justified as follows:

$$\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\left[Q^{k/M}\right]^{B\alpha-D\gamma(n+1)}\right] = \sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\left[Q^{k/M}\right]^{M[\alpha/\beta-\gamma(n+1)/\delta]}\right] \quad (160)$$

$$= \sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\left[Q^{[\alpha/\beta-\gamma(n+1)/\delta]}\right]^k\right] \quad (161)$$

$$= \sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k-1)/2}}\left[Q^{[\alpha/\beta-\gamma(n+1)/\delta-1]}\right]^k\right] \quad (162)$$

$$= \theta\left(Q;-Q^{\alpha/\beta-(n+1)\gamma/\delta-1}\right), \quad (163)$$

where (160) follows from $B = M/\beta$, $D = M/\delta$ as in (96); (161) follows from cancelling out M; (162) follows from multiplying up and down by Q^{-k}; and (163) follows from (10). □

Examining (159), one sees that if $B\alpha - D\gamma(n+1) \neq 0 \mod M$ in (159), then by Lemma 6, one has $\sum_{\ell=0}^{M-1}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma(n+1)} = 0$ and (159) vanishes. On the other hand, if $B\alpha - D\gamma(n+1) = 0 \mod M$, then there is a $k \in \mathbb{Z}$ with

$$Mk = B\alpha - D\gamma(n+1) = M\left[\frac{\alpha}{\beta}-\frac{\gamma}{\delta}(n+1)\right] \Longleftrightarrow k = \left[\frac{\alpha}{\beta}-\frac{\gamma}{\delta}(n+1)\right] \Longleftrightarrow k-1 = \left[\frac{\alpha}{\beta}-\frac{\gamma}{\delta}(n+1)-1\right] \in \mathbb{Z}, \quad (164)$$

whence $\theta(Q;-Q^{\alpha/\beta-(n+1)\gamma/\delta-1}) = 0$ by (13). Thus, (159) again vanishes. Since (159) now vanishes in all cases, we have vanishing of every derivative of $\mathcal{F}[\mathscr{W}_{\mu,\lambda}(t)](x)$ at $x=0$, giving

(155). Thus, (154) now holds, and we have vanishing of all moments. This completes the proof of the proposition.

Notice the similarity of the argument in Proposition 20 with that of Proposition 18. In each case, a factor formed from the sum of powers of roots of unity fails to vanish precisely when the remaining theta function factor vanishes.

The next proposition gives a simple characterization for $\mathscr{W}_{\mu,\lambda}(t)$ to vanish (and to not vanish) identically.

Proposition 21. *For all notation as in (96), the following equivalences hold:*

$$[\delta \neq 0 \mod \beta] \Leftrightarrow \mathscr{W}_{\mu,\lambda}(t) \equiv 0 \Leftrightarrow \frac{1}{M}\mathscr{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x) \equiv 0 \tag{165}$$

$$\Leftrightarrow \frac{1}{M}\left[\sum_{\ell=0}^{M-1}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{\left[\omega^{\ell+1}\right]^{D\gamma}}\right)\right] \equiv 0 \tag{166}$$

$$\Leftrightarrow \frac{\mu_Q^3 e^{i\pi\alpha/\beta}}{\sqrt{2\pi}}\frac{1}{(-ix)}\left(\frac{1}{D\gamma}\left[\sum_{\kappa=0}^{D\gamma-1}\frac{\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right]\right) \equiv 0. \tag{167}$$

Thus, $[\delta = 0 \mod \beta]$ occurs precisely when $\mathscr{W}_{\mu,\lambda}(t)$ does not vanish identically, which in turn occurs precisely when the identity (98) and (99) in Theorem 13 is not an identically zero tautology.

Proof. The right-most equivalence in (165) holds by linearity and injectivity of the Fourier transform; and the equivalences in (166) and (167) hold by Proposition 19 and Theorem 13, respectively. It remains to show the following equivalence:

$$[\delta \neq 0 \mod \beta] \Leftrightarrow [\text{the vanishing of } (167)]. \tag{168}$$

Observe that in the argument of the theta function in (167), the expression $[\tilde{\omega}^{\kappa}]^M = [\tilde{\omega}^{k}]^M$ holds precisely when $\exists\, j$ with $e^{2\pi ij/D}\tilde{\omega}^{\kappa} = \tilde{\omega}^{k}$ (and then $k = \kappa + j\gamma$). One might expect that the previous statement would be that $\exists\, J$ with $e^{2\pi iJ/M}\tilde{\omega}^{\kappa} = \tilde{\omega}^{k}$; however, $e^{2\pi iJ/M} = e^{2\pi iJ/[D\delta]} = e^{2\pi iJ\gamma/[(D\gamma)\delta]}$ is an integral power of $\tilde{\omega} = e^{2\pi i/[D\gamma]}$ precisely when $J = j\delta$ is a multiple of δ, resulting in canceling of δ terms. □

Next, summing over indices with like values of $[\tilde{\omega}^{\kappa}]^M$ first gives

$$\begin{aligned}
&\left[\sum_{\kappa=0}^{D\gamma-1}\frac{\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right] \\
&\quad = \sum_{\kappa=0}^{\gamma-1}\left[\sum_{j=0}^{D-1}\frac{\left[e^{2\pi ij/D}\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[e^{2\pi ij/D}\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right] \\
&\quad = \sum_{\kappa=0}^{\gamma-1}\left[\sum_{j=0}^{D-1}\frac{\left[e^{2\pi ij/D}\right]^{B\alpha}\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right] \\
&\quad = \sum_{\kappa=0}^{\gamma-1}\left[\left(\sum_{j=0}^{D-1}\left[e^{2\pi ij/D}\right]^{B\alpha}\right)\frac{\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right]
\end{aligned} \tag{169}$$

$$= \left[\sum_{j=0}^{D-1}\left[e^{2\pi j/D}\right]^{B\alpha}\right]\left[\sum_{\kappa=0}^{\gamma-1}\left[\frac{\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right]\right]. \tag{170}$$

By Proposition 22 below, the expression $\sum_{\kappa=0}^{\gamma-1}[[\tilde{\omega}^{\kappa}z_3]^{B\alpha}/\theta(Q;[\tilde{\omega}^{\kappa}z_3]^{M})]$ in (170) does not vanish identically. Thus, (169) and (170) vanish identically if and only if $\sum_{j=0}^{D-1}[e^{2\pi j/D}]^{B\alpha}$ vanishes. However, this summation of D^{th} roots of unity to the $B\alpha$ power vanishes precisely when $B\alpha \neq 0 \mod D$ by Lemma 6. Thus, $B\alpha \neq 0 \mod D$ is equivalent to identically vanishing in (169) and (170) and therefore equivalent to identically vanishing in (167). Now,

$$\begin{aligned}
&B\alpha \neq 0 \mod D \Longleftrightarrow \nexists\, n \text{ such that } B\alpha \\
&\quad = Dn \Longleftrightarrow \nexists\, n \text{ such that } \frac{M}{\beta}\alpha \\
&\quad = \frac{M}{\delta}n \Longleftrightarrow \nexists\, n \text{ such that } \alpha\frac{\delta}{\beta} \\
&\quad = n \Leftrightarrow \delta \neq 0 \mod \beta,
\end{aligned} \tag{171}$$

where the last equivalence holds from the fact that α/β is in a reduced form. Thus, the leftmost equivalence in (165) holds. From this left-most equivalence, one deduces that $\mathscr{W}_{\mu,\lambda}(t)$ does not vanish identically precisely when $\delta = 0 \mod \beta$. The proposition is now proven.

The following proposition, utilized in Proposition 21, relies on properties of the Jacobi theta function to obtain a nonidentically vanishing condition.

Proposition 22. *The function*

$$\sum_{\kappa=0}^{\gamma-1}\left[\frac{\left[\tilde{\omega}^{\kappa}z\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z\right]^{M}\right)}\right] \tag{172}$$

is not identically 0 in the argument $z \in \mathbb{C}$. Hence,

$$\sum_{\kappa=0}^{\gamma-1}\left[\frac{\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right] \tag{173}$$

given in (170) is not identically 0 in z_3, where z_3 is as in (100) and (184).

Proof. If $\gamma = 1$, then (172) becomes $[z]^{B\alpha}/\theta(Q;z^M)$, which does not vanish identically. For instance when $z = 1$, it becomes $1/\theta(Q;1) \neq 0$. If $\gamma > 1$, we show that the function

$$\sum_{\kappa=0}^{\gamma-1}\left[\frac{[\tilde{\omega}^{\kappa}z]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z]^{M}\right)}\right] = \frac{\sum_{\kappa=0}^{\gamma-1}\left([\tilde{\omega}^{\kappa}z]^{B\alpha}\prod_{0\le j\ne\kappa}^{\gamma-1}\theta\left(Q;[\tilde{\omega}^{j}z]^{M}\right)\right)}{\prod_{j=0}^{\gamma-1}\theta\left(Q;[\tilde{\omega}^{j}z]^{M}\right)} \tag{174}$$

is also not identically 0. This in turn is equivalent to the numerator in the right hand side of (174), namely,

$$\sum_{\kappa=0}^{\gamma-1}\left([\tilde{\omega}^{\kappa}z]^{B\alpha}\prod_{0\le j\ne\kappa}^{\gamma-1}\theta\left(Q;[\tilde{\omega}^{j}z]^{M}\right)\right) = z^{B\alpha}\theta\left(Q;[\tilde{\omega}z]^{M}\right)\theta\left(Q;[\tilde{\omega}^{2}z]^{M}\right)\cdots\theta\left(Q;[\tilde{\omega}^{\gamma-1}z]^{M}\right) \tag{175}$$

$$+\theta\left(Q;[z]^{M}\right)\sum_{\kappa=1}^{\gamma-1}\left([\tilde{\omega}^{\kappa}z]^{B\alpha}\prod_{1\le j\ne\kappa}^{\gamma-1}\theta\left(Q;[\tilde{\omega}^{j}z]^{M}\right)\right), \tag{176}$$

not being identically 0 in z. Setting $z = (-Q)^{1/M}$ in (175) and (176) gives that

$$\theta\left(Q;\left[(-Q)^{1/M}\right]^{M}\right) = \theta(Q;-Q) = 0 \tag{177}$$

and then only the summand in (175) survives:

$$\begin{aligned}\sum_{\kappa=0}^{\gamma-1}&\left(\left[\tilde{\omega}^{\kappa}(-Q)^{1/M}\right]^{B\alpha}\prod_{0\le j\ne\kappa}^{\gamma-1}\theta\left(Q;[\tilde{\omega}^{j}]^{M}[-Q]\right)\right)\\ &= \left[(-Q)^{1/M}\right]^{B\alpha}\theta\left(Q;[\tilde{\omega}]^{M}(-Q)\right)\theta\\ &\cdot\left(Q;[\tilde{\omega}^{2}]^{M}(-Q)\right)\cdots\theta\left(Q;[\tilde{\omega}^{\gamma-1}]^{M}(-Q)\right)\neq 0,\end{aligned} \tag{178}$$

where the nonvanishing in (178) is obtained from (13) along with the fact that

$$-Q[\tilde{\omega}^{j}]^{M} = -Q\left[e^{2\pi ij/[D\gamma]}\right]^{M} = -Q\left[e^{2\pi ij\delta/\gamma}\right] \tag{179}$$

does not lie on the negative real axis for $j = 1, \cdots, \gamma-1$. To see this latter point, if $j\delta/\gamma = n \in \mathbb{Z}$, one has $j\delta = \gamma n$, from which one has that j is divisible by γ (as γ/δ is assumed to be in reduced form). However, each $j = 1, \cdots, \gamma-1$ is not divisible by γ. Thus, (172) does not vanish identically. By the identity theorem, (172) does not vanish on any subset of $\mathbb{C}$ having a limit point. One concludes that (173) is not identically 0 in z_3, as the set of z_3 as in (184) ranges over a set with limit point as x varies in $\mathbb{R}$. The proposition is now shown. □

In light of Proposition 21 and Lemma 14, we refine and sharpen Theorem 13 by

(1) removing all identically $0 = 0$ tautologies in (98) and (99) via making the additional assumption that $\delta = 0 \bmod \beta$

(2) guaranteeing that the expressions in (98) and (99) are well-defined via making the easily checked assumption that $\delta \neq 0 \bmod 4$

(3) incorporating $\mathscr{W}_{\mu,\lambda}(t)$ into the theorem while including the additional properties of $\mathscr{W}_{\mu,\lambda}(t)$ garnered from Propositions 17–20

In doing so, we arrive at the main theorem of this study.

Theorem 23. *Let $q > 1$, and for $t \ge 0$, let $f_{\mu,\lambda}(t)$ be defined as in (1), with $\mu \in \mathbb{Q}$, $\lambda \in \mathbb{Q}^{+}$. For $t \in \mathbb{R}$, let $\mathscr{W}_{\mu,\lambda}(t)$, as in Definition 16, be the function naturally generated by $f_{\mu,\lambda}(t)$. Let the notation of (96) hold. In particular, let $(\mu+1)/2 = \alpha/\beta$ and $\lambda/2 = \gamma/\delta$ be in a reduced form; let $\delta \neq 0 \bmod 4$ with $\delta = 0 \bmod \beta$ (which gives $M = \delta$ and $D = 1$); and let $\omega = e^{2\pi i/\delta}$ with $\tilde{\omega} = e^{2\pi i/\gamma}$. Then, $\mathscr{W}_{\mu,\lambda}(t)$ is a real valued Schwartz wavelet with all moments vanishing with Fourier transform given by $\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)$ satisfying*

$$\frac{1}{\delta}\mathscr{F}\left[\mathscr{W}_{\mu,\lambda}(t)\right](x) \tag{180}$$

$$= \frac{1}{\delta}\left[\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](x) + \sum_{\ell=0}^{\delta-2}\left[\omega^{\ell+1}\right]^{B\alpha-\gamma}\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{[\omega^{\ell+1}]^{\gamma}}\right)\right] \tag{181}$$

$$= \frac{\mu_{Q}^{3}e^{i\pi\alpha/\beta}}{\sqrt{2\pi}}\frac{1}{(-ix)}\left(\frac{1}{\gamma}\left[\sum_{\kappa=0}^{\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^{\delta}\right)}\right]\right), \tag{182}$$

where $\theta(Q;z)$ is the Jacobi theta function, where

$$Q = q^{\delta/\gamma}, \quad B = \frac{\delta}{\beta}, \quad \omega = e^{2\pi i/\delta}, \quad \tilde{\omega} = e^{2\pi i/\gamma}, \tag{183}$$

and where for z_2 any fixed γ^{th} root of ix, one has that z_3 in (182) is given by

$$z_3 = e^{i\pi/\gamma}e^{-i\pi/\delta}z_2, \quad z_3^{\gamma} = -e^{-\pi i\gamma/\delta}ix. \tag{184}$$

Also, $\mathscr{F}[\tilde{f}_{\mu,\lambda}(t)](x/[\omega^{\ell+1}]^{D\gamma})$ is given by (17). Furthermore, given $f_{\mu,\lambda}(t)$, one has that $\mathscr{W}_{\mu,\lambda}(t)$ is uniquely defined by Definition 16, and it satisfies the same multiplicatively advanced differential equation (MADE) on $\mathbb{R}$ as does $f_{\mu,\lambda}(t)$ on $[0,\infty)$, namely,

$$\mathscr{W}_{\mu,\lambda}^{(\delta)}(t) = (-1)^{\gamma+\delta}q^{\gamma(\gamma+\mu)/\lambda}\mathscr{W}_{\mu,\lambda}(q^{\gamma}t). \tag{185}$$

Proof. $\mathscr{W}_{\mu,\lambda}(t)$ is real valued and Schwartz via Propositions 17 and 18. From Proposition 19, one has that equality in (187) below holds, and, from equation (98) and (99) in Theorem 13, one has that equality in (188) below holds.

$$\frac{1}{M}\mathcal{F}\left[\mathcal{W}_{\mu,\lambda}(t)\right](x) \tag{186}$$

$$= \frac{1}{M}\left[\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](x) + \sum_{\ell=0}^{M-2}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma} \cdot \mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{\left[\omega^{\ell+1}\right]^{D\gamma}}\right)\right] \tag{187}$$

$$= \frac{\mu_Q^3 e^{i\pi\alpha/\beta}}{\sqrt{2\pi}}\frac{1}{(-ix)}\left(\frac{1}{D\gamma}\left[\sum_{\kappa=0}^{D\gamma-1}\frac{\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right]\right). \tag{188}$$

From Lemma 14, the hypothesis that $\delta \neq 0 \bmod 4$ is equivalent to the condition that (186)–(188) are defined for all $x \in \mathbb{R}$. By Proposition 21, $\mathcal{W}_{\mu,\lambda}(t)$ is identically 0 in t precisely when $\delta \neq 0 \bmod \beta$; thus, the hypothesis that $\delta = 0 \bmod \beta$ in the theorem gives precisely all the nonidentically zero examples of (186)–(188). Furthermore, from the hypothesis that $\delta = 0 \bmod \beta$, one has that $M = \operatorname{lcm}\{\beta, \delta\} = \delta$, from which we conclude that $D = M/\delta = \delta/\delta = 1$. Setting $M = \delta$ and $D = 1$ in (186)–(188) yields equations (180)–(182). Similarly, setting $M = \delta$ and $D = 1$ in (96) (respectively (100)) yields (183) (respectively (184)). Next, all moments vanish by Proposition 20. It remains to show that $\mathcal{W}_{\mu,\lambda}(t)$ is a wavelet solving the MADE (185). The wavelet property follows from the following three criteria:

(1) $\mathcal{W}_{\mu,\lambda}(t) \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$ because $\mathcal{W}_{\mu,\lambda}(t)$ is Schwartz

(2) $\int_{-\infty}^{\infty}\mathcal{W}_{\mu,\lambda}(t)\,dt = 0$ because each moment, including the 0^{th}-moment, of $\mathcal{W}_{\mu,\lambda}(t)$ vanishes

(3) $\int_{-\infty}^{\infty}((|\mathcal{F}[\mathcal{W}_{\mu,\lambda}(t)](x)|^2)/|x|)\,dx < \infty$ because $\mathcal{F}[\mathcal{W}_{\mu,\lambda}(t)](x)$ is Schwartz and decays rapidly in the tails and because $\mathcal{F}[\mathcal{W}_{\mu,\lambda}(t)](x)$ vanishes to infinite order at $x = 0$ by Proposition 20

□

Now, from [2], $f_{\mu,\lambda}(t)$ on $[0, \infty)$ satisfies the MADE

$$f_{\mu,\lambda}^{(\delta)}(t) = (-1)^{\gamma+\delta}q^{\gamma(\gamma+\mu)/\lambda}f_{\mu,\lambda}(q^{\gamma}t). \tag{189}$$

This follows, since

$$\frac{\lambda}{2} = \frac{\gamma}{\delta}, \quad \text{whence} \quad \lambda = \frac{2\gamma}{\delta}. \tag{190}$$

Setting $A = \gamma$ and $L = \delta$ in equations (16)–(18) in Theorem 2.2 of [2] gives the MADE (189). The MADE for $\mathcal{W}_{\mu,\lambda}(t)$ now follows from (189) as $\mathcal{W}_{\mu,\lambda}(t)$ is a linear combination of expressions involving $f_{\mu,\lambda}(t)$ (by Definition 16 and by (125)–(127)). This proves the theorem.

Remark 24. While the assumption that $\delta = 0 \bmod \beta$ in Theorem 23 may at first seem to be limiting, in fact, there remains a wealth of nonidentically-zero cases (with $\delta = 0 \bmod \beta$). The examples below in Sections 5 and 6 demonstrate that Theorem 23 is quite general in nature. For now, observe that if δ is a multiple of β then $\operatorname{lcm}\{\beta, \delta\} = \delta = M$. Hence, if $M = \delta = \prod_{j=1}^{J}p_j^{n_j}$ is the prime factorization of M, the nonidentically zero cases for $\mathcal{W}_{\mu,\lambda}(t)$ and its Fourier transform are handled by those $\beta = \prod_{j=1}^{J}p_j^{k_j}$ with $0 \le k_j \le n_j$ for all $j \in \{1, \cdots, J\}$ and with the parameters μ and λ satisfying $(\mu+1)/2 = \alpha/\beta$ and $\lambda/2 = \gamma/\delta$ in a reduced form.

Remark 25. We also point out that even when $\delta \neq 0 \bmod \beta$, we are able to relate the Fourier transform of $f_{\mu,\lambda}(t)$ to the Jacobi theta function. However, the Fourier relation in this setting comes from an analogue of $\mathcal{W}_{\mu,\lambda}(t)$ that is generically noncontinuous at $t = 0$ and is not a wavelet. See Theorem 28 and the related discussion below.

Remark 26. In a development similar to (169)-(170), it is also possible to show that

$$\left[\sum_{\ell=0}^{M-1}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{\left[\omega^{\ell+1}\right]^{D\gamma}}\right)\right] \tag{191}$$

$$= \left[\sum_{j=0}^{D-1}\left[e^{2\pi j/D}\right]^{B\alpha}\right]\sum_{\ell=0}^{\delta-1}\left[\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{\left[\omega^{\ell+1}\right]^{D\gamma}}\right)\right]. \tag{192}$$

However, the development in (169) and (170) is preferred in order to harness properties of the Jacobi theta function to gain a nonidentically vanishing criterion, as in Proposition 22. Nonetheless, in comparing the expression in (170) with that in (192), observing that they share a common factor $[\sum_{j=0}^{D-1}[e^{2\pi j/D}]^{B\alpha}]$, and contemplating equation (98) - (99), one is led to consider the relation between

$$\sum_{\kappa=0}^{\gamma-1}\left[\frac{\left[\tilde{\omega}^{\kappa}z_3\right]^{B\alpha}}{\theta\left(Q;\left[\tilde{\omega}^{\kappa}z_3\right]^{M}\right)}\right] \quad \text{and} \quad \sum_{\ell=0}^{\delta-1}\left[\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathcal{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{\left[\omega^{\ell+1}\right]^{D\gamma}}\right)\right], \tag{193}$$

including when $\delta \neq 0 \bmod \beta$. This will be done in Theorem 28 below, where it will allow for the recovery of Fourier transform information for $f_{\mu,\lambda}(t)$ in the cases that $\delta \neq 0 \bmod \beta$.

Remark 27. At this juncture, we pause to convey a surprising consequence of expressing the Fourier transform of $\mathcal{W}_{\mu,\lambda}$ in terms of Jacobi theta functions, as in Theorem 23. Knowledge of the Fourier transform of $\mathcal{W}_{\mu,\lambda}$ in terms of the Jacobi theta function allows for the proof of nonvanishing results in the cases of $\gamma = 1, 2$ in Corollary 34 and Proposition 36 below. This pair of nonvanishing results serves to underpin a main connection between the solutions of MADEs of type $\mathscr{W}_{\mu,\lambda}$ to the theory of wavelet frames in harmonic analysis. That is, in these cases, $\mathscr{W}_{\mu,\lambda}$ is a mother wavelet generating a wavelet frame for the square integrable functions $\mathscr{L}^2(\mathbb{R})$, as is demonstrated in Theorem 37 below. Thus, the current study lays the foundation for a set of strong connections between solutions of MADEs, special function theory, and the theory of wavelet frames. Next, we recover information on the Fourier transform of $\tilde{f}_{\mu,\lambda}(t)$ when $\delta \neq 0 \bmod \beta$ in the second main theorem of this work, again relating the Fourier transform to the Jacobi theta function.

Theorem 28. *For $q > 1$, let $f_{\mu,\lambda}(t)$ be as in (1) and $\tilde{f}_{\mu,\lambda}(t)$ be as in (14), respectively, where we assume $\mu \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}^+$, with notation as in (96), in particular $(\mu + 1)/2 = \alpha/\beta$ and $\lambda/2 = \gamma/\delta$ are in reduced form. Let $\theta(Q; z)$ be the Jacobi theta function. Let $\delta \neq 0 \bmod \beta$ and let $\delta \neq 0 \bmod 4$. Then,*

$$\frac{1}{\mu_Q^3}\frac{1}{\delta}\sum_{\ell=-1}^{\delta-2}\sum_{k=-\infty}^{\infty}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[\omega^{\ell+1}Q^{k/M}\right]^{B\alpha}}{ix+\left[\omega^{\ell+1}Q^{k/M}\right]^{D\gamma}}\right] \tag{194}$$

$$= \frac{e^{i\pi\alpha/\beta}}{(-ix)}\left(\frac{1}{\gamma}\sum_{\kappa=\kappa_1}^{\kappa_1+\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^{M}\right)}\right) \tag{195}$$

$$+\frac{D\left[e^{2\pi i\alpha\delta/\beta}-1\right]}{2\pi i}\int_{R_1}\frac{1}{\nu\theta(Q;\nu^M)}\frac{\nu^{B\alpha}e^{i\pi\alpha/\beta}}{\left(ix+\nu^{D\gamma}e^{i\pi\gamma/\delta}\right)}\,d\nu. \tag{196}$$

Here, for z_2 a fixed $[D\gamma]^{th}$ root of ix, one has that z_3 in (195) (and (201) below) is given by

$$z_3 = e^{i\pi/[D\gamma]}e^{-i\pi\gamma/[\delta D\gamma]}z_2, \quad \text{whereby} \quad z_3^{D\gamma} = -e^{-\pi i\gamma/\delta}ix. \tag{197}$$

Also, R_1 is the ray emanating from the origin given by (210) and (206), and

$$\kappa_1 \equiv \min\left\{\kappa \mid \arg\ (\tilde{\omega}^{\kappa}z_3) > -\frac{\pi}{M}\right\}. \tag{198}$$

Furthermore, μ_Q is given by (11), with

$$\omega = e^{2\pi i/M} \text{ and } \tilde{\omega} = e^{2\pi i/[D\gamma]}, \tag{199}$$

where M is taken to be the least common multiple of β and δ.

One also has the following relation of the weighted partial average of the rotations of the Fourier transforms $\mathscr{F}[\tilde{f}_{\mu,\lambda}(t)](x)$ with a partial average of rotations of $z_3/\theta(Q; z_3^M)$:

$$\frac{1}{\delta}\left[\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right](x)+\sum_{\ell=0}^{\delta-2}\left[\omega^{\ell+1}\right]^{B\alpha-D\gamma}\mathscr{F}\left[\tilde{f}_{\mu,\lambda}(t)\right]\left(\frac{x}{[\omega^{\ell+1}]^{D\gamma}}\right)\right] \tag{200}$$

$$= \frac{\mu_Q^3\, e^{i\pi\alpha/\beta}}{\sqrt{2\pi}}\frac{1}{(-ix)}\left(\frac{1}{\gamma}\left[\sum_{\kappa=\kappa_1}^{\kappa_1+\gamma-1}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q;[\tilde{\omega}^{\kappa}z_3]^{M}\right)}\right]\right) \tag{201}$$

$$+\frac{\mu_Q^3\, D\left[e^{2\pi i\alpha\delta/\beta}-1\right]}{[2\pi]^{3/2}i}\int_{R_1}\frac{1}{\nu\theta(Q;\nu^M)}\frac{\nu^{B\alpha}e^{i\pi\alpha/\beta}}{ix+\nu^{D\gamma}e^{i\pi\gamma/\delta}}\,d\nu. \tag{202}$$

Again, z_3 in (201) is given by (197). The expression in (202) is referred to as the defect.

Proof. Instead of integrating the integrand $f_2(\nu)$ in (64) over the oriented boundary Γ_N of the annulus A_N as in Theorems 13 and 23, we shall integrate $f_2(\nu)$ over the oriented boundary $\tilde{\Gamma}_N$ of the intersection of the sector $S = S(\phi_1, \phi_2) = \{\nu \in \mathbb{C}^* \mid \phi_1 \le \arg(\nu) \le \phi_2\} \cup \{0\}$ with the annulus A_N, where ϕ_1 is chosen as in (206) below and $\phi_2 - \phi_1 = 2\pi/D$. Thus, $\tilde{\Gamma}_N = R_{1,N} + S_N - R_{2,N} - s_N$. Here, $s_N \equiv S \cap c_N$ and $S_N \equiv S \cap C_N$, where as before, $C_N = e^{i\phi}[Q^{(N+1)/M} + Q^{N/M}]/2$ and $c_N = e^{i\phi}[Q^{(-N-1)/M} + Q^{-N/M}]/2$, but in the current setting, one has S_N and s_N that ϕ increases from ϕ_1 to ϕ_2. Also, $R_{1,N}$ and $R_{2,N}$ are portions of two rays, as given by

$$R_{1,N} = \left\{\nu = \rho e^{i\phi_1}\ \middle|\ \frac{Q^{(-N-1)/M}+Q^{-N/M}}{2} \le \rho \le \frac{Q^{(N+1)/M}+Q^{N/M}}{2}\right\}, \tag{203}$$

$$R_{2,N} = \left\{\nu = \rho e^{i\phi_2}\ \middle|\ \frac{Q^{(-N-1)/M}+Q^{-N/M}}{2} \le \rho \le \frac{Q^{(N+1)/M}+Q^{N/M}}{2}\right\}. \tag{204}$$

□

Referring to (117) and (118) in Lemma 14, one has

$$\delta \neq 0 \bmod 4 \Longleftrightarrow \forall\ x \in \mathbb{R}\quad ix \notin \tilde{S} = \left\{-\left[\omega^{j+1}Q^{k/M}\right]^{D\gamma}\ \middle|\ j,k\in\mathbb{Z}\right\}, \tag{205}$$

and this in turn is equivalent to the fact that for all $x \in \mathbb{R}$ the roots (in ν) of $ix + [\nu]^{D\gamma}e^{i\pi\gamma/\delta} = 0$ do not lie among the zeroes of $\theta(Q; \nu^M)$ by the remarks at the ends of Propositions 8 and 9. Hence, for all $x \in \mathbb{R}$ and each associated z_3 as in (197) and for each integer κ, one has $\tilde{\omega}^{\kappa}z_3$ which does not fall among the zeroes of $\theta(Q; \nu^M)$. Namely, for all $x \in \mathbb{R}$ and each associated z_3, and for all values of κ one has $\tilde{\omega}^{\kappa}z_3 \notin \{e^{\pi i/M}\omega^{\ell}Q^{k/M} \mid \ell, k \in \mathbb{Z}\}$. In particular, setting $\ell = -1$, then, for all $x \in \mathbb{R}$ and each associated z_3 and for all values of κ and n, one has that $\arg\ (\tilde{\omega}^{\kappa}z_3) \neq -\pi/M + 2\pi n$. We now choose

$$\phi_1 \equiv -\pi/M - \varepsilon, \quad \phi_2 \equiv \phi_1 + 2\pi/D, \tag{206}$$

where $\varepsilon < 2\pi/M$ is chosen sufficiently small so that none of the $\tilde{\omega}^{\kappa}z_3$ falls in the sector $S(-\pi/M - \varepsilon, -\pi/M)$. Then, the corresponding sector $S = S(\phi_1, \phi_2)$ gives the intersection S

$\cap A_N$ which has oriented boundary $\widetilde{\Gamma}_N = R_{1,N} + S_N - R_{2,N} - s_N$ as describe above. Define $\kappa_1 \equiv \min\{\kappa \mid \arg(\tilde{\omega}^\kappa z_3) > -\pi/M\}$. Choose N sufficiently large such that all $\tilde{\omega}^\kappa z_3$ are contained in A_N. Then integrating $f_2(v)$ in (64) over the contour $\widetilde{\Gamma}_N$ gives $2\pi i$ times the enclosed residues. Namely,

$$\int_{\widetilde{\Gamma}_N} f_2(v)\,dv = \int_{\widetilde{\Gamma}_N} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv = 2\pi i \sum_{\ell=-1}^{\delta-2} \sum_{k=-N}^{N} Res\left(f_2, \omega^\ell e^{i\pi/M} Q^{k/M}\right) + 2\pi i \sum_{\kappa=\kappa_1}^{\kappa_1+\gamma-1} Res(f_2, \tilde{\omega}^\kappa z_3) \quad (207)$$

$$= 2\pi i \frac{1}{\mu_Q^3} \frac{1}{M} \sum_{\ell=-1}^{\delta-2} \sum_{k=-N}^{N} \left[\frac{(-1)^{k+1}}{Q^{k(k+1)/2}} \frac{\left[\omega^{\ell+1} Q^{k/M}\right]^{B\alpha}}{ix + \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}} \right] \quad (208)$$

$$+2\pi i \frac{e^{i\pi\alpha/\beta}}{(-ix)} \frac{1}{D\gamma} \sum_{\kappa=\kappa_1}^{\kappa_1+\gamma-1} \frac{[\tilde{\omega}^\kappa z_3]^{B\alpha}}{\theta\left(Q; [\tilde{\omega}^\kappa z_3]^M\right)}, \quad (209)$$

where the evaluation of the enclosed residues in (207) occurs via Propositions 8 and 9, similar to the analogous computation in the setting of Theorem 13. Also, comparing (208) with (75), notice the summation (over ℓ) in (208) has $M/D = \delta$ terms (as compared to M terms in (75)) as the sector $S = S(\phi_1, \phi_2)$ sweeps through an argument of $2\pi/D$ in (208) as opposed to sweeping through an argument of 2π in (75). The choice of $\phi_1 = -\pi/M - \varepsilon$ is made in order to have ℓ start at -1 in (208), whereby the first term (at $\ell = -1$) in the double summation in (208) will have $\omega^{\ell+1} = \omega^0 = 1$, matching the expression for $\mathscr{F}[\tilde{f}_{\mu,\lambda}](x)$ in (16) after letting N approach infinity. Again, comparing (209) with (76), notice the summation (over κ) in (209) has $[D\gamma]/D = \gamma$ terms (as compared to $D\gamma$ terms in (76)) as the sector $S = S(\phi_1, \phi_2)$ sweeps through an argument of $2\pi/D$ in (209) as opposed to sweeping through an argument of 2π in (76). The choice of κ_1 is made to give $\tilde{\omega}^{\kappa_1} z_3$ as the first (v) root of $ix + v^{D\gamma} e^{\pi i\gamma/\delta} = 0$ falling in $S(\phi_1, \phi_2) \cap A_N$, that is with smallest argument such that $\phi_1 < \arg(\tilde{\omega}^\kappa z_3) < \phi_2$.

Now, defining the rays

$$R_1 = \left\{ v = \rho e^{i\phi_1} \mid 0 \le \rho < \infty \right\}, \quad (210)$$

$$R_2 = \left\{ v = \rho e^{i\phi_2} \mid 0 \le \rho < \infty \right\}, \quad (211)$$

one has that integration along $R_{m,N}$ approaches integration along R_m for $m = 1, 2$ as $N \longrightarrow \infty$. Hence, one has the following limit:

$$\lim_{N\longrightarrow\infty} \int_{\widetilde{\Gamma}_N} f_2(v)\,dv = \lim_{N\longrightarrow\infty} \int_{\widetilde{\Gamma}_N} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv = \lim_{N\longrightarrow\infty} \int_{R_{1,N}+S_N-R_{2,N}-s_N} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv = \lim_{N\longrightarrow\infty} \int_{R_{1,N}} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv \quad (212)$$

$$- \lim_{N\longrightarrow\infty} \int_{R_{2,N}} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv = \int_{R_1} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv \quad (213)$$

$$- \int_{R_2} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv = \int_{R_1} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv \quad (214)$$

$$- \int_{R_1} \frac{1}{v e^{2\pi i/D} \theta\left(Q; [v e^{2\pi i/D}]^M\right)} \frac{\left[v e^{2\pi i/D}\right]^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + [v e^{2\pi i/D}]^{D\gamma} e^{i\pi\gamma/\delta}\right)} e^{2\pi i/D} dv \quad (215)$$

$$= \int_{R_1} \left[\frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{v\theta(Q; v^M)\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} - \frac{\left[v e^{2\pi i/D}\right]^{B\alpha} e^{i\pi\alpha/\beta}}{v\theta(Q; v^M)\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} \right] dv \quad (216)$$

$$= \left[1 - e^{2\pi i\alpha\delta/\beta}\right] \int_{R_1} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv, \quad (217)$$

where in moving from (212) to (213), one can drop the integrals over S_N and s_N in the limit by Corollary 12; in moving from (213) to (214), we have relied on the remarks following (211); in moving from (214) to (215) we used $R_2 = e^{2\pi i/D} R_1$; in moving from (215) to (216), we have simplified powers of $e^{2\pi i/D}$ by utilizing $M/D = \delta$ and $D\gamma/D = \gamma$; and in moving from (216) to (217), we have used $B = M/\beta$ and $D = M/\delta$. Relying on (217) and taking the limit of (207)–(209) as N approaches infinity yield

$$\left[1 - e^{2\pi i\alpha\delta/\beta}\right] \int_{R_1} \frac{1}{v\theta(Q; v^M)} \frac{v^{B\alpha} e^{i\pi\alpha/\beta}}{\left(ix + v^{D\gamma} e^{i\pi\gamma/\delta}\right)} dv = 2\pi i \frac{1}{\mu_Q^3} \frac{1}{M} \sum_{\ell=-1}^{\delta-2} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^{k+1}}{Q^{k(k+1)/2}} \frac{\left[\omega^{\ell+1} Q^{k/M}\right]^{B\alpha}}{ix + \left[\omega^{\ell+1} Q^{k/M}\right]^{D\gamma}} \right] \quad (218)$$

$$+2\pi i \frac{e^{i\pi\alpha/\beta}}{(-ix)} \frac{1}{D\gamma} \sum_{\kappa=\kappa_1}^{\kappa_1+\gamma-1} \frac{[\tilde{\omega}^\kappa z_3]^{B\alpha}}{\theta\left(Q; [\tilde{\omega}^\kappa z_3]^M\right)}. \quad (219)$$

Rearranging terms in (218) and (219) and multiplying through by $D/[2\pi i]$ give (194)–(196). Referring to (16), multiplying (194)-(196) by $\mu_Q^3/\sqrt{2\pi}$, and factoring out the powers $[\omega^{\ell+1}]^{B\alpha}$ from the numerator and $[\omega^{\ell+1}]^{D\gamma}$ from the denominator of (194) give (200)–(202). The theorem is now demonstrated.

Remark 29. When $x = 0$, we will consider the expression (202) (the integral together with its coefficient) to be the defect from having the expression in (200) vanish at $x = 0$, as the expression in (201) vanishes as x approaches 0 by Corollary 32 below. As a consequence, one concludes that if the defect (202) (when $x = 0$) does not vanish, then expression (200) cannot be the Fourier transform of a wavelet. Observe that when $\delta \neq 0 \bmod \beta$, the term $[e^{2\pi i\alpha\delta/\beta} - 1]$ in (202) does not vanish, and vanishing of the defect when $x = 0$ is equivalent to the vanishing of the integral expression in (202).

Remark 30. If one were to allow for the case $\delta = 0 \bmod \beta$ in Theorem 28, one would then have $M = \operatorname{lcm}\{\beta, \delta\} = \delta$ and $D = M/\delta = 1$ in this case. Furthermore, in this case, Theorem 28 would reduce to Theorem 23, as the coefficient would be $[e^{2\pi i\alpha\delta/\beta} - 1] = [1 - 1] = 0$ in (202). Thus, if one were to remove the hypothesis that $\delta \neq 0 \bmod \beta$ in Theorem 28, Theorem 23 could be subsumed into Theorem 28. We choose to keep the cases $\delta = 0 \bmod \beta$ and $\delta \neq 0 \bmod \beta$ separate in order to highlight the special properties of $\mathscr{W}_{\mu,\lambda}(t)$ when $\delta = 0 \bmod \beta$.

The following proposition gives a pair of flatness conditions, and it forms the technical basis for the proof of Corollary 32.

Proposition 31. *Let $q > 1$ and ϕ be fixed with $-\pi < \phi < \pi$. Then for $0 < r < \infty$ and for any power $p \in \mathbb{R}$, one has*

$$\lim_{r\longrightarrow 0} \frac{r^p}{\theta(q\,;re^{\phi})} = 0 = \lim_{r\longrightarrow\infty} \frac{r^p}{\theta(q\,;re^{\phi})}. \qquad (220)$$

Due to its length, we leave the proof of Proposition 31 until the end of this section, where the interested reader can peruse it. Instead, we immediately proceed to Corollary 32 and its proof, as Corollary 32 is used throughout this study.

Corollary 32. *Let $\delta \neq 0 \bmod 4$, and let z_3 be as in (184) and (197). Namely, $z_3 = e^{i\pi/[D\gamma]}e^{-i\pi\gamma/[\delta D\gamma]}z_2$, where z_2 is any $[D\gamma]^{th}$ root of ix. As a consequence, $z_3^{D\gamma} = -e^{-\pi i\gamma/\delta}ix$. Let $\tilde{\omega} = e^{2\pi i/[D\gamma]}$. Then for $\kappa = 1, \cdots, D\gamma$*

$$\lim_{x\longrightarrow 0}\left(\frac{1}{ix}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z_3]^M\right)}\right) = 0. \qquad (221)$$

Proof. Note that $-e^{\pi i\gamma/\delta}z_3^{D\gamma} = ix$. Hence,

$$\begin{aligned}\frac{1}{ix}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z_3]^M\right)} &= \frac{1}{\left(-e^{\pi i\gamma/\delta}z_3^{D\gamma}\right)}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z_3]^M\right)}\\ &= -e^{-\pi i\gamma/\delta}[\tilde{\omega}^{\kappa}]^{D\gamma}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha-D\gamma}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z_3]^M\right)}.\end{aligned} \qquad (222)$$

One concludes that

$$\left|\frac{1}{ix}\frac{[\tilde{\omega}^{\kappa}z_3]^{B\alpha}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z_3]^M\right)}\right| = \frac{\left[|z_3|^M\right]^{(B\alpha-D\gamma)/M}}{\left|\theta\left(Q\,;[\tilde{\omega}^{\kappa}z_3]^M\right)\right|}. \qquad (223)$$

Applying Proposition 31 with $p = (B\alpha - D\gamma)/M$ as $x \longrightarrow 0$ gives (221). Note that the assumption that $\delta \neq 0 \bmod 4$ assures that $[\tilde{\omega}^{\kappa}z_3]^M$ does not fall along the negative real axis. This gives the corollary. □

We finish this section with the proof of Proposition 31.

Proof of Proposition 31. Set $r = q^{\tau}$, where τ ranges from $-\infty$ to ∞. Then, from (10), one has

$$\begin{aligned}\left|\theta\left(q\,;re^{i\phi}\right)\right|^2 &= \left|\theta\left(q\,;q^{\tau}e^{i\phi}\right)\right|^2 = \theta\left(q\,;q^{\tau}e^{i\phi}\right)\theta\left(q\,;q^{\tau}e^{-i\phi}\right)\\ &= \mu_q\prod_{n=0}^{\infty}\left(1+\frac{q^{\tau}e^{i\phi}}{q^n}\right)\left(1+\frac{1}{q^{\tau}e^{i\phi}q^{n+1}}\right)\\ &\cdot\mu_q\prod_{n=0}^{\infty}\left(1+\frac{q^{\tau}e^{-i\phi}}{q^n}\right)\left(1+\frac{1}{q^{\tau}e^{-i\phi}q^{n+1}}\right)\\ &= \left(\mu_q\right)^2\prod_{n=0}^{\infty}\left(1+\frac{2q^{\tau}\cos(\phi)}{q^n}+\frac{q^{2\tau}}{q^{2n}}\right)\\ &\cdot\prod_{n=0}^{\infty}\left(1+\frac{2\cos(\phi)}{q^{\tau}q^{n+1}}+\frac{1}{q^{2\tau}q^{2(n+1)}}\right)\\ &= \left(\mu_q\right)^2\prod_{n=0}^{\infty}\left(\left[q^{\tau-n}+\cos(\phi)\right]^2+\sin^2(\phi)\right)\\ &\cdot\prod_{n=0}^{\infty}\left(\left[q^{-\tau-n-1}+\cos(\phi)\right]^2+\sin^2(\phi)\right)\end{aligned} \qquad (224)$$

$$= \left(\mu_q\right)^2 J(\tau)J(-\tau-1), \qquad (225)$$

where

$$J(\tau) \equiv \prod_{n=0}^{\infty}\left(\left[q^{\tau-n}+\cos(\phi)\right]^2+\sin^2(\phi)\right). \qquad (226)$$

From (224) and (10), one has that, for $\cos(\phi) \geq 0$,

$$\left|\theta\left(q\,;q^{\tau}e^{i\phi}\right)\right|^{2} \geq \left(\mu_q\right)^2 \prod_{n=0}^{\infty}\left(1+\frac{q^{2\tau}}{q^{2n}}\right)\prod_{n=0}^{\infty}\left(1+\frac{1}{q^{2\tau}q^{2(n+1)}}\right) = \frac{\left(\mu_q\right)^2}{\mu_{q^2}}\theta\left(q^2\,;q^{2\tau}\right) = \frac{\left(\mu_q\right)^2}{\mu_{q^2}}\sum_{m\in\mathbb{Z}}\frac{\left(q^{2\tau}\right)^m}{\left(q^2\right)^{m(m-1)/2}} \tag{227}$$

$$> \frac{\left(\mu_q\right)^2}{\mu_{q^2}}\frac{\left(q^{2\tau}\right)^m}{\left(q^2\right)^{m(m-1)/2}}, \tag{228}$$

where the inequality in (228) holds for all $m \in \mathbb{Z}$. Hence, inverting (227) and (228), taking square roots, and multiplying by $r^p = q^{\tau p}$, one obtains that for each $m \in \mathbb{Z}$

$$\left|\frac{r^p}{\theta\left(q\,;re^{i\phi}\right)}\right| = \left|\frac{q^{\tau p}}{\theta\left(q\,;q^{\tau}e^{i\phi}\right)}\right| \leq \frac{\sqrt{\mu_{q^2}}}{\mu_q}q^{m(m-1)/2}q^{(p-m)\tau}. \tag{229}$$

To handle the limit as $\tau \longrightarrow +\infty$, pick $m > p$. To handle the limit as $\tau \longrightarrow -\infty$, pick $m < p$. Thus, (220) holds when $\cos(\phi) \geq 0$.

The remaining case is $-1 < \cos(\phi) < 0$. Here, the possibility that $\cos(\phi) = -1$ is excluded, since if $\cos(\phi)$ were to equal -1 then $\theta(q\,;re^{i\phi}) = \theta(q;-r)$, which has an infinite number of zeroes (for $r = q^j$ with $j \in \mathbb{Z}$). It will be shown below (see (239) through (241)) that we have the following two limits:

$$\lim_{\tau\longrightarrow\infty} J(-\tau-1) = 1 = \lim_{\tau\longrightarrow-\infty} J(\tau). \tag{230}$$

It will also be shown (see (242) and (259) below) that the following rate of growth holds:

$$J(\tau) \approx q^{2\tau\lfloor\tau\rfloor-\lfloor\tau\rfloor^2+\lfloor\tau\rfloor} = q^{\lfloor\tau\rfloor^2+\{1+2(\tau-\lfloor\tau\rfloor)\}\lfloor\tau\rfloor} \text{ as } \tau \longrightarrow \infty, \tag{231}$$

where $\lfloor\tau\rfloor$ denotes the greatest integer function evaluated at τ. Hence, from (231), we have the related rate of growth

$$J(-\tau-1) \approx q^{2(-\tau-1)\lfloor-\tau-1\rfloor-\lfloor-\tau-1\rfloor^2+\lfloor-\tau-1\rfloor} = q^{\lfloor|\tau|\rfloor^2+\{-1+2(|\tau|-\lfloor|\tau|\rfloor)\}\lfloor|\tau|\rfloor-2[|\tau|-\lfloor|\tau|\rfloor]} \text{ as } \tau \longrightarrow -\infty. \tag{232}$$

For the time being, we assume (230)–(232) and use (225) to show that (220) holds as $r \longrightarrow \infty$ (equivalently as $\tau \longrightarrow \infty$) via

$$\left|\frac{r^p}{\theta\left(q\,;re^{i\phi}\right)}\right| = \left|\frac{q^{\tau p}}{\theta\left(q\,;q^{\tau}e^{i\phi}\right)}\right| = \frac{q^{p\tau}}{\mu_q\sqrt{J(\tau)J(-\tau-1)}} \approx \frac{q^{p\tau}}{q^{\{\lfloor\tau\rfloor+1+2(\tau-\lfloor\tau\rfloor)\}\lfloor\tau\rfloor/2}} \tag{233}$$

$$= q^{p(\tau-\lfloor\tau\rfloor+\lfloor\tau\rfloor)-\{\lfloor\tau\rfloor+1+2(\tau-\lfloor\tau\rfloor)\}\lfloor\tau\rfloor/2} = q^{\{2p-\lfloor\tau\rfloor-1-2(\tau-\lfloor\tau\rfloor)\}\lfloor\tau\rfloor/2}q^{p(\tau-\lfloor\tau\rfloor)} \longrightarrow 0 \text{ as } \tau \longrightarrow \infty, \tag{234}$$

where one moves to (233) via (230) and (231). The vanishing in (234) is seen from the fact that when $\lfloor\tau\rfloor > 2p-1$ the exponent $\{2p-\lfloor\tau\rfloor-1-2(\tau-\lfloor\tau\rfloor)\}\lfloor\tau\rfloor/2$ in (234) is increasingly negative as $\tau \longrightarrow \infty$, while $p(\tau-\lfloor\tau\rfloor)$ remains bounded.

Similarly, we use (225) to show next that (220) holds as $r \longrightarrow 0$ (equivalently as $\tau \longrightarrow -\infty$). In this setting,

$$\left|\frac{r^p}{\theta\left(q\,;re^{i\phi}\right)}\right| = \left|\frac{q^{\tau p}}{\theta\left(q\,;q^{\tau}e^{i\phi}\right)}\right| = \frac{q^{p\tau}}{\mu_q\sqrt{J(\tau)J(-\tau-1)}} \approx \frac{q^{-p(-\tau)}}{q^{\{\lfloor|\tau|\rfloor-1+2(|\tau|-\lfloor|\tau|\rfloor)\}\lfloor|\tau|\rfloor/2-[|\tau|-\lfloor|\tau|\rfloor]}} \tag{235}$$

$$= q^{-p(|\tau|-\lfloor|\tau|\rfloor+\lfloor|\tau|\rfloor)-\{\lfloor|\tau|\rfloor-1+2(|\tau|-\lfloor|\tau|\rfloor)\}\lfloor|\tau|\rfloor/2+[|\tau|-\lfloor|\tau|\rfloor]} \tag{236}$$

$$= q^{-\{2p+\lfloor|\tau|\rfloor-1+2(|\tau|-\lfloor|\tau|\rfloor)\}\lfloor|\tau|\rfloor/2+(-p+1)[|\tau|-\lfloor|\tau|\rfloor]} \longrightarrow 0 \text{ as } \tau \longrightarrow -\infty. \tag{237}$$

where one moves to (235) and (236) via (230) and (232). The vanishing in (237) is seen from the fact that when $\lfloor|\tau|\rfloor > 1-2p$ the exponent $-\{2p+\lfloor|\tau|\rfloor-1+2(|\tau|-\lfloor|\tau|\rfloor)\}\lfloor|\tau|\rfloor/2$ in (237) is increasingly negative as $\tau \longrightarrow -\infty$, while $(-p+1)[|\tau|-\lfloor|\tau|\rfloor]$ remains bounded.

All that remains is to show (230)–(232). We first show $\lim_{\tau\longrightarrow\infty} J(-\tau-1) = 1$, from which it will immediately follow that $\lim_{\tau\longrightarrow-\infty} J(\tau) = 1$. Note that

$$\begin{aligned} J(-\tau-1) &= \prod_{n=0}^{\infty}\left(\left[q^{-\tau-n-1}+\cos(\phi)\right]^2+\sin^2(\phi)\right) \\ &= \prod_{n=0}^{\infty}\left(\frac{1}{q^{2(\tau+n+1)}}+\frac{2\cos(\phi)}{q^{\tau+n+1}}+1\right) \\ &= \prod_{n=0}^{\infty}\left(\frac{1}{q^{2(\tau-\lfloor\tau\rfloor+\lfloor\tau\rfloor+n+1)}}+\frac{2\cos(\phi)}{q^{\tau-\lfloor\tau\rfloor+\lfloor\tau\rfloor+n+1}}+1\right) \\ &= \prod_{k=\lfloor\tau\rfloor+1}^{\infty}\left(\frac{1}{q^{2(\tau-\lfloor\tau\rfloor)+2k}}+\frac{2\cos(\phi)}{q^{(\tau-\lfloor\tau\rfloor)+k}}+1\right), \end{aligned} \tag{238}$$

where the reindexing $k = \lfloor\tau\rfloor+n+1$ occurs in moving to (238). For τ satisfying $\lfloor\tau\rfloor > \log_q(2)$ and $-1 < \cos(\phi) < 0$, one traps $J(-\tau-1)$ in (238) by

$$\exp\left(\sum_{k=\lfloor\tau\rfloor+1}^{\infty}\frac{-4}{q^k}\right) < \prod_{k=\lfloor\tau\rfloor+1}^{\infty}\left(\frac{-2}{q^k}+1\right) < J(-\tau-1) < \prod_{k=\lfloor\tau\rfloor+1}^{\infty}\left(\frac{1}{q^{2k}}+1\right) < \exp\left(\sum_{k=\lfloor\tau\rfloor+1}^{\infty}\frac{1}{q^{2k}}\right), \tag{239}$$

where we have further assumed that τ is sufficiently large so that $\ln\left(1-2/q^{\lfloor\tau\rfloor+1}\right) > -4/q^{\lfloor\tau\rfloor+1}$, (that is $1-2/q^{\lfloor\tau\rfloor+1}$ falls in the interval $\{x \in \mathbb{R} \mid \ln(x) > 2(x-1)\} \supset (2/3, 1) \neq \emptyset$). Since

$$\lim_{\tau \longrightarrow \infty} \exp\left(\sum_{k=\lfloor\tau\rfloor+1}^{\infty} \frac{-4}{q^k}\right) = \lim_{\tau \longrightarrow \infty} \exp\left(\frac{-4}{q^{\lfloor\tau\rfloor+1}} \frac{q}{(q-1)}\right) = 1, \tag{240}$$

$$\lim_{\tau \longrightarrow \infty} \exp\left(\sum_{k=\lfloor\tau\rfloor+1}^{\infty} \frac{1}{q^{2k}}\right) = \lim_{\tau \longrightarrow \infty} \exp\left(\frac{1}{q^{2(\lfloor\tau\rfloor+1)}} \frac{q^2}{(q^2-1)}\right) = 1 \tag{241}$$

hold, the trapping in (239) gives that $\lim_{\tau \longrightarrow \infty} J(-\tau-1) = 1$. An immediate consequence is that $\lim_{\tau \longrightarrow -\infty} J(\tau) = 1$, giving (230). We conclude that there is an N_1 so that for $\tau > N_1$ one has

$$\frac{1}{2} \leq J(-\tau-1) \leq \frac{3}{2}. \tag{242}$$

We next show (231). From (226), observe that for $\tau \geq 1$ one has

$$\begin{aligned} J(\tau) &= \prod_{n=0}^{\infty} \left([q^{\tau-n} + \cos(\phi)]^2 + \sin^2(\phi) \right) \\ &= \prod_{n=0}^{\lfloor\tau\rfloor-1} \left([q^{\tau-n} + \cos(\phi)]^2 + \sin^2(\phi) \right) \\ &\quad \cdot \prod_{n=\lfloor\tau\rfloor}^{\infty} \left([q^{\tau-n} + \cos(\phi)]^2 + \sin^2(\phi) \right) \\ &= \prod_{n=0}^{\lfloor\tau\rfloor-1} \left([q^{\tau-n} + \cos(\phi)]^2 + \sin^2(\phi) \right) \\ &\quad \cdot \prod_{n=\lfloor\tau\rfloor}^{\infty} \left(\left[q^{\tau-\lfloor\tau\rfloor-(n-\lfloor\tau\rfloor)} + \cos(\phi)\right]^2 + \sin^2(\phi) \right) \end{aligned} \tag{243}$$

$$\begin{aligned} &= \prod_{n=0}^{\lfloor\tau\rfloor-1} \left([q^{\tau-n} + \cos(\phi)]^2 + \sin^2(\phi) \right) \\ &\quad \cdot \prod_{k=0}^{\infty} \left(\left[q^{\tau-\lfloor\tau\rfloor-k} + \cos(\phi)\right]^2 + \sin^2(\phi) \right) \end{aligned} \tag{244}$$

$$\begin{aligned} &= \prod_{n=0}^{\lfloor\tau\rfloor-1} \left([q^{\tau-n} + \cos(\phi)]^2 + \sin^2(\phi) \right) J(\tau - \lfloor\tau\rfloor) \\ &= \prod_{n=0}^{\lfloor\tau\rfloor-1} q^{2(\tau-n)} \prod_{n=0}^{\lfloor\tau\rfloor-1} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-n}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-n}}\right]^2 \right) J(\tau - \lfloor\tau\rfloor) \\ &= q^{2\tau\lfloor\tau\rfloor} \left(q^{-2}\right)^{\lfloor\tau\rfloor(\lfloor\tau\rfloor-1)/2} \prod_{n=0}^{\lfloor\tau\rfloor-1} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-n}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-n}}\right]^2 \right) \\ &\quad \cdot J(\tau - \lfloor\tau\rfloor) \end{aligned} \tag{245}$$

$$\begin{aligned} &= q^{2\tau\lfloor\tau\rfloor - \lfloor\tau\rfloor^2 + \lfloor\tau\rfloor} \prod_{n=0}^{\lfloor\tau\rfloor-1} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-n}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-n}}\right]^2 \right) \\ &\quad \cdot J(\tau - \lfloor\tau\rfloor), \end{aligned} \tag{246}$$

where a reindexing by $k = n - \lfloor\tau\rfloor$ moves one from (243) to (244); one obtains (245) from (244) by (226); and one factors out a $q^{2(\tau-n)}$ from each product in the left most product expression in (245) to proceed forward.

Now, $J(\tau)$ is nonvanishing for all τ, as $|\theta(q; re^{i\phi})|^2 = |\theta(q; q^{\tau} e^{i\phi})|^2$ has vanishing points if and only if $\phi = \pm\pi$, up to full revolutions, and $\tau \in \mathbb{Z}$ (by (10)). Thus, the compact interval $[0, 1]$ under J maps to a compact interval $[M_1, M_2]$ (with $0 < M_1 \leq M_2 < \infty$). Since $\tau - \lfloor\tau\rfloor \in [0, 1)$, one has

$$0 < M_1 \leq J(\tau - \lfloor\tau\rfloor) \leq M_2 < \infty, \tag{247}$$

and $J(\tau - \lfloor\tau\rfloor)$ is bounded for all τ.

It remains to show that $\prod_{n=0}^{\lfloor\tau\rfloor-1}([1 + \cos(\phi)/q^{\tau-n}]^2 + [\sin(\phi)/q^{\tau-n}]^2)$ in (246) is bounded as τ approaches infinity (again, in the setting that $-1 < \cos(\phi) < 0$). An upper bound C_2 independent of τ follows via

$$\begin{aligned} &\prod_{n=0}^{\lfloor\tau\rfloor-1} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-n}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-n}}\right]^2 \right) \\ &\quad = \prod_{n=0}^{\lfloor\tau\rfloor-1} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-\lfloor\tau\rfloor+(\lfloor\tau\rfloor-n)}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-\lfloor\tau\rfloor+(\lfloor\tau\rfloor-n)}}\right]^2 \right) \end{aligned} \tag{248}$$

$$= \prod_{k=1}^{\lfloor\tau\rfloor} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-\lfloor\tau\rfloor+k}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-\lfloor\tau\rfloor+k}}\right]^2 \right) \tag{249}$$

$$\begin{aligned} &= \prod_{k=1}^{\lfloor\tau\rfloor} \left(1 + \frac{2\cos(\phi)}{q^{\tau-\lfloor\tau\rfloor+k}} + \frac{1}{q^{2(\tau-\lfloor\tau\rfloor)+2k}} \right) \\ &< \prod_{k=1}^{\lfloor\tau\rfloor} \left(1 + \frac{1}{q^{2k}} \right) \leq \exp\left(\sum_{k=1}^{\lfloor\tau\rfloor} \frac{1}{q^{2k}} \right) \end{aligned} \tag{250}$$

$$\leq \exp\left(\sum_{k=1}^{\infty} \frac{1}{q^{2k}} \right) = \exp\left(\frac{1}{q^2-1} \right) \equiv C_2, \tag{251}$$

where the reindexing $k = \lfloor\tau\rfloor - n$ is used to obtain (249).

Starting with (248)–(249) to obtain (252), a lower bound C_1 independent of τ for $\tau \geq N$ (with N determined below) follows via

$$\begin{aligned} &\prod_{n=0}^{\lfloor\tau\rfloor-1} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-n}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-n}}\right]^2 \right) \\ &\quad = \prod_{k=1}^{\lfloor\tau\rfloor} \left(\left[1 + \frac{\cos(\phi)}{q^{\tau-\lfloor\tau\rfloor+k}}\right]^2 + \left[\frac{\sin(\phi)}{q^{\tau-\lfloor\tau\rfloor+k}}\right]^2 \right) \end{aligned} \tag{252}$$

$$> \prod_{k=1}^{\lfloor\tau\rfloor}\left(\left[1+\frac{\cos(\phi)}{q^k}\right]^2+\left[\frac{\sin(\phi)}{q^{1+k}}\right]^2\right) = \prod_{k=1}^{\lfloor\tau\rfloor}\left(1+\frac{2\cos(\phi)}{q^k}+\frac{\cos^2(\phi)}{q^{2k}}+\frac{\sin^2(\phi)}{q^{2(k+1)}}\right) \quad (253)$$

$$= \prod_{k=1}^{N-1}\left(1+\frac{2\cos(\phi)}{q^k}+\frac{\cos^2(\phi)}{q^{2k}}+\frac{\sin^2(\phi)}{q^{2(k+1)}}\right) \cdot \prod_{k=N}^{\lfloor\tau\rfloor}\left(1+\frac{2\cos(\phi)}{q^k}+\frac{\cos^2(\phi)}{q^{2k}}+\frac{\sin^2(\phi)}{q^{2(k+1)}}\right) \quad (254)$$

$$> \prod_{k=1}^{N-1}\left(1+\frac{2\cos(\phi)}{q^k}+\frac{\cos^2(\phi)}{q^{2k}}\right)\prod_{k=N}^{\lfloor\tau\rfloor}\left(1+\frac{2\cos(\phi)}{q^k}\right) \quad (255)$$

$$> \prod_{k=1}^{N-1}\left(1+\frac{\cos(\phi)}{q^k}\right)^2 \exp\left(\sum_{k=N}^{\lfloor\tau\rfloor}\frac{4\cos(\phi)}{q^k}\right) \quad (256)$$

$$> \left(1+\frac{\cos(\phi)}{q}\right)^{2(N-1)} \exp\left(\frac{4\cos(\phi)}{q^N}\frac{q}{(q-1)}\right) \equiv C_1, \quad (257)$$

where N appearing in (254) is chosen sufficiently large so that $\ln(1+2\cos(\phi)/q^N) > 4\cos(\phi)/q^N$ (that is, $1+2\cos(\phi)/q^N$ falls in the interval $\{x\in\mathbb{R} \mid \ln(x) > 2(x-1)\} \supset (2/3,1) \neq \varnothing$). This justifies movement from (255) to (256).

Hence, we have from (251) and (257) that

$$0 < C_1 < \prod_{n=0}^{\lfloor\tau\rfloor-1}\left(\left[1+\frac{\cos(\phi)}{q^{\tau-n}}\right]^2+\left[\frac{\sin(\phi)}{q^{\tau-n}}\right]^2\right) < C_2. \quad (258)$$

From (242), (247), and (258) applied to (225) and (246), one obtains

$$\begin{aligned} 0 &< \left(\mu_q\right)^2 q^{2\tau\lfloor\tau\rfloor-\lfloor\tau\rfloor^2+\lfloor\tau\rfloor} C_1 M_1 \frac{1}{2} \\ &< \left(\mu_q\right)^2 J(\tau)J(-\tau-1) = \left|\theta\left(q;q^\tau e^{i\phi}\right)\right|^2 \\ &\le \left(\mu_q\right)^2 q^{2\tau\lfloor\tau\rfloor-\lfloor\tau\rfloor^2+\lfloor\tau\rfloor} C_2 M_2 \frac{3}{2} < \infty, \end{aligned} \quad (259)$$

for $\tau > \max\{N_1, N\}$, where $q > 1$ is fixed. Thus, (231) is demonstrated. As a consequence, we also have that (232) now holds. The proposition is proven.

4. Connection to the Theory of Wavelet Frames

We have seen in the previous section that expressing the Fourier transform of $\mathscr{W}_{\mu,\lambda}(t)$ in terms of Jacobi theta functions, as in Theorem 23, provides a strong connection to the theory of special functions. However, more can be concluded. Namely, from the relation of the $\mathscr{W}_{\mu,\lambda}(t)$ to the Jacobi theta function, we also demonstrate in this section the connection to the theory of wavelets and wavelet frames. In particular, for low values of γ, we establish that each $\mathscr{W}_{\mu,\lambda}(t)$ as in Theorem 23 is a Schwartz mother wavelet for a wavelet frame generating all of $\mathscr{L}^2(\mathbb{R})$.

Recall that $f(t)$ is a wavelet if it belongs to $\mathscr{L}^1(\mathbb{R}) \cap \mathscr{L}^2(\mathbb{R}) \cap \mathscr{L}^\infty(\mathbb{R})$, has first moment $\int_{-\infty}^{\infty} f(t)\,dt = 0$, and satisfies the admissibility condition that $\int_{-\infty}^{\infty} |\mathscr{F}[f(t)](x)|^2/|x|\,dx < \infty$. Furthermore, such a wavelet $f(t)$ is a mother wavelet for a frame of form

$$S(f;a_0,b_0) = \left\{a_0^{n/2} f(a_0^n t + m b_0)/\|f\| \,\middle|\, n, m \in \mathbb{Z}\right\}, \quad (260)$$

if $S(f;a_0,b_0)$ generates $\mathscr{L}^2(\mathbb{R})$, where $a_0 > 1$ is the scale factor, $b_0 > 0$ is the translation parameter, and $\|f\| = \|f\|_2$ is the norm of f in $\mathscr{L}^2(\mathbb{R})$. One defines the diagonal term $G_0[f](x)$ by

$$G_0[f](x) = \frac{1}{\|f\|^2}\sum_{n=-\infty}^{\infty}\left|\mathscr{F}[f(t)](a_0^n x)\right|^2 \quad (261)$$

and the off-diagonal term $G_1[f](x)$ by

$$G_1[f](x) = \frac{1}{\|f\|^2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}\setminus\{0\}}\left|\mathscr{F}[f(t)]\left(a_0^j x\right) \cdot \mathscr{F}[f(t)]\left(a_0^j x + 2\pi k/b_0\right)\right|, \quad (262)$$

which together give the frame condition

$$\begin{aligned} 0 &< \inf_{1\le|x|\le a_0}\{G_0[f](x) - G_1[f](x)\} \\ &\le \sup_{1\le|x|\le a_0}[G_0[f](x) - G_1[f](x)] < \infty, \end{aligned} \quad (263)$$

sufficient for $S(f;a_0,b_0)$ in (260) to be a frame. As is shown in [36] and [37], for $b_0 > 0$ sufficiently small, (263) is in turn implied by the conditions (264) immediately below:

$$0 < \inf_{1\le|x|\le a_0}\{G_0[f](x)\} \quad \text{and} \quad \exists\, C > 0 \quad \text{with} \quad |\mathscr{F}[f(t)](x)| \le \frac{C|x|}{(1+x^2)^{3/2}}. \quad (264)$$

In Proposition 36 below, we see that there are natural scale factors a_0 that allow us to compute $G_0[f](x)$ for the wavelet $f(t) = \mathscr{W}_{\mu,\lambda}(t)$ for low values of γ, where knowledge of properties of Jacobi theta functions lets us compute $G_0[\mathscr{W}_{\mu,\lambda}](x)$ exactly, which in turn will establish the nonvanishing of $G_0[\mathscr{W}_{\mu,\lambda}](x)$ in the left hand criteria of (264).

But first, we need to obtain an even stronger nonvanishing analogue of Proposition 22 in the setting of Theorem 23 (in particular, with $\delta = 0 \bmod \beta$) under the additional assumption that $\gamma = 1, 2$.

Proposition 33. *Under the notation and assumptions of Theorem 23 (in particular $\delta = 0 \mod \beta$ and $M = \delta$), along with the additional assumption that $\gamma = 1, 2$, one has that (172) becomes*

$$\sum_{\kappa=0}^{\gamma-1}\left[\frac{[\tilde{\omega}^{\kappa}z]^{B\alpha}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z]^{\delta}\right)}\right], \tag{265}$$

which never vanishes for $z \in \mathbb{C}^*$ if $\gamma = 1$ and for $\gamma = 2$ and $z \in \mathbb{C}^*$ (265) vanishes precisely when

$$\begin{cases} z^{2\delta} = -Q^{4p+1+2j}, & \text{for some } p \in \mathbb{Z} \text{ with } j = 0 \quad \text{if } B\alpha \text{ is even}, \\ z^{2\delta} = -Q^{4p+1+2j}, & \text{for some } p \in \mathbb{Z} \text{ with } j = 1 \quad \text{if } B\alpha \text{ is odd}. \end{cases} \tag{266}$$

Equivalently, (265) vanishes precisely when $z = e^{\pi i/[2\delta]} e^{2\pi i \ell/[2\delta]} Q^{[4p+1+2j]/[2\delta]}$ for some $p, \ell \in \mathbb{Z}$, where $j = 0$ if $B\alpha$ is even, and $j = 1$ if $B\alpha$ is odd.

Proof. If $\gamma = 1$, then $\tilde{\omega} = e^{2\pi i/\gamma} = e^{2\pi i/1} = 1$ and (265) becomes $[z]^{B\alpha}/\theta(Q\,;[z]^{\delta})$, which never vanishes for $z \in \mathbb{C}^*$. If $\gamma = 2$, then $\tilde{\omega} = e^{2\pi i/\gamma} = e^{2\pi i/2} = -1$ and (265) becomes

$$\begin{aligned}\sum_{\kappa=0}^{\gamma-1}\left[\frac{[\tilde{\omega}^{\kappa}z]^{B\alpha}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z]^{\delta}\right)}\right] &= \frac{[z]^{B\alpha}}{\theta\left(Q\,;[z]^{\delta}\right)} + \frac{[-z]^{B\alpha}}{\theta\left(Q\,;[-z]^{\delta}\right)} \\ &= z^{B\alpha}\left[\frac{[1]^{B\alpha}\theta(Q;-z^{\delta}) + [-1]^{B\alpha}\theta(Q\,;z^{\delta})}{\theta(Q\,;z^{\delta})\theta(Q;-z^{\delta})}\right],\end{aligned} \tag{267}$$

where equality in (267) follows from the fact that δ must be odd, as γ/δ is in reduced form and $\gamma = 2$. Now from (10) the numerator in the bracketed expression in (267) becomes

$$[1]^{B\alpha}\theta\left(Q;-z^{\delta}\right) + [-1]^{B\alpha}\theta\left(Q\,;z^{\delta}\right) \tag{268}$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{\left(-z^{\delta}\right)^{n}}{Q^{n(n-1)/2}} + [-1]^{B\alpha}\frac{\left(z^{\delta}\right)^{n}}{Q^{n(n-1)/2}}\right] \tag{269}$$

$$= \sum_{n=-\infty}^{\infty}\frac{\left(-z^{\delta}\right)^{n} + [-1]^{B\alpha}\left(z^{\delta}\right)^{n}}{Q^{n(n-1)/2}} \tag{270}$$

$$= \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty}\frac{2\left(z^{\delta}\right)^{2k}}{Q^{2k(2k-1)/2}}, & \text{if } B\alpha \text{ is even}, \\ \displaystyle\sum_{k=-\infty}^{\infty} -\frac{2\left(z^{\delta}\right)^{2k+1}}{Q^{(2k+1)(2k)/2}}, & \text{if } B\alpha \text{ is odd}, \end{cases} \tag{271}$$

where

(1) if $B\alpha$ is even in (270) the $n =$ odd cases cancel, leaving the $n = 2k$ in the $B\alpha$ even case of (271)

(2) if $B\alpha$ is odd in (270) the $n =$ even cases cancel, leaving the $n = 2k+1$ in the $B\alpha$ odd case of (271)

Now, in the $B\alpha$ even case,

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{2\left(z^{\delta}\right)^{2k}}{Q^{2k(2k-1)/2}} &= \sum_{k=-\infty}^{\infty}\frac{2\left(z^{2\delta}\right)^{k}Q^{-k}}{Q^{\left(4k^2-2k\right)/2}Q^{-2k/2}} \\ &= 2\sum_{k=-\infty}^{\infty}\frac{\left(z^{2\delta}/Q\right)^{k}}{\left[Q^{4}\right]^{k(k-1)/2}} = 2\theta\left(Q^{4}\,;\frac{z^{2\delta}}{Q}\right),\end{aligned} \tag{272}$$

where the last equality in (272) follows from (10). And in the $B\alpha$ odd case,

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{-2\left(z^{\delta}\right)^{2k+1}}{Q^{(2k+1)(2k)/2}} &= \sum_{k=-\infty}^{\infty}\frac{-2z^{\delta}\left(z^{2\delta}\right)^{k}Q^{-3k}}{Q^{\left(4k^2+2k\right)/2}Q^{-6k/2}} \\ &= -2z^{\delta}\sum_{k=-\infty}^{\infty}\frac{\left(z^{2\delta}/Q^{3}\right)^{k}}{\left[Q^{4}\right]^{k(k-1)/2}} = -2z^{\delta}\theta\left(Q^{4}\,;\frac{z^{2\delta}}{Q^{3}}\right),\end{aligned} \tag{273}$$

where the last equality in (273) again follows from (10). Thus, (268)–(271) reduces to

$$\begin{aligned}&[1]^{B\alpha}\theta\left(Q;-z^{\delta}\right) + [-1]^{B\alpha}\theta\left(Q\,;z^{\delta}\right) \\ &\quad= \begin{cases} 2\theta\left(Q^{4}\,;\dfrac{z^{2\delta}}{Q}\right), & \text{if } B\alpha \text{ is even}, \\ -2z^{\delta}\theta\left(Q^{4}\,;\dfrac{z^{2\delta}}{Q^{3}}\right), & \text{if } B\alpha \text{ is odd}. \end{cases}\end{aligned} \tag{274}$$

Thus, (267) reduces to

$$\begin{aligned}&\sum_{\kappa=0}^{\gamma-1}\left[\frac{[\tilde{\omega}^{\kappa}z]^{B\alpha}}{\theta\left(Q\,;[\tilde{\omega}^{\kappa}z]^{\delta}\right)}\right] \\ &\quad= \frac{[z]^{B\alpha}}{\theta\left(Q\,;[z]^{\delta}\right)} + \frac{[-z]^{B\alpha}}{\theta\left(Q\,;[-z]^{\delta}\right)} \\ &\quad= \begin{cases} 2z^{B\alpha}\left[\dfrac{\theta(Q^{4}\,;z^{2\delta}/Q)}{\theta(Q\,;z^{\delta})\theta(Q;-z^{\delta})}\right], & \text{if } B\alpha \text{ is even} \\ -2z^{B\alpha+\delta}\left[\dfrac{\theta(Q^{4}\,;(z^{2\delta}/Q^{3}))}{\theta(Q\,;z^{\delta})\theta(Q;-z^{\delta})}\right], & \text{if } B\alpha \text{ is odd} \end{cases}\end{aligned} \tag{275}$$

$$= (-1)^{j}\,2\,z^{B\alpha+j\delta}\left[\frac{\theta(Q^{4}\,;(z^{2\delta}/Q^{1+2j}))}{\theta(Q\,;z^{\delta})\theta(Q;-z^{\delta})}\right], \quad \text{where} \begin{cases} j = 0, & \text{if } B\alpha \text{ is even}, \\ j = 1, & \text{if } B\alpha \text{ is odd}. \end{cases} \tag{276}$$

Then, (276) vanishes for $z \in \mathbb{C}^*$ precisely when $\theta(Q^{4}\,;z^{2\delta}/Q^{1+2j}) = 0$, which by (10) occurs precisely when $z^{2\delta}/$

$Q^{1+2j} = -Q^{4p}$ for some $p \in \mathbb{Z}$. This last statement is equivalent to (266). Equivalently, we have such vanishing when $z = (-1)^{1/[2\delta]} Q^{[4p+1+2j]/[2\delta]} = e^{\pi i/[2\delta]} e^{2\pi i \ell/[2\delta]} Q^{[4p+1+2j]/[2\delta]}$ for some $p, \ell \in \mathbb{Z}$. This gives the proposition. □

Corollary 34. *In the setting of, and under the assumptions of, Theorem 23, with $\delta = 0 \bmod \beta$ and $\delta \neq 0 \bmod 4$, with $\gamma = 1$ or $\gamma = 2$, one has that*

$$\sum_{\kappa=0}^{\gamma-1} \left[\frac{[\tilde{\omega}^\kappa z_3]^{B\alpha}}{\theta\left(Q ; [\tilde{\omega}^\kappa z_3]^\delta\right)} \right] \neq 0, \quad \text{for all } x \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}, \tag{277}$$

where $z_3 = e^{\pi i/\gamma} e^{-\pi i/\delta} z_2$ and z_2 is any fixed γ^{th} root of ix.

Proof. By Proposition 33, when $\gamma = 1$, the nonvanishing of (277) holds automatically for $x \neq 0$. Also by Proposition 33, equation (266), when $\gamma = 2$, the vanishing of (277) holds if and only if for

$$z_3 = e^{\pi i/\gamma} e^{-\pi i/\delta} e^{\pi i/(2\gamma)} e^{i\nu/\gamma} |x|^{1/\gamma} \tag{278}$$

$$= e^{\pi i/2} e^{-\pi i/\delta} e^{\pi i/4} e^{i\nu/2} |x|^{1/2}, \tag{279}$$

one has $z_3^{2\delta} = -Q^{4p+1+2j}$ for some $p \in \mathbb{Z}$ and for $j = 0$ if $B\alpha$ is even or for $j = 1$ if $B\alpha$ is odd, where in (278)–(279) one has $\nu = 0$ if $x > 0$ and $\nu = \pi$ if $x < 0$. From (279) and the fact that δ is odd (since $\gamma = 2$ is even), one sees that

$$z_3^{2\delta} = e^{i3\pi\delta/2} e^{i\nu\delta} |x|^\delta = (\pm i)(\pm 1)|x|^\delta = \pm i|x|^\delta \neq -Q^{4p+1+2j}, \tag{280}$$

for any values of p and j, because $\pm i|x|^\delta$ is imaginary while $-Q^{4p+1+2j}$ is real. Thus, (277) never vanishes for $x \in \mathbb{R}^*$ when $\gamma = 2$, and the corollary is proven. □

Remark 35. Although, for $\gamma = 1, 2$ by Corollary 34 one has (277) never vanishes for $x \in \mathbb{R}^*$, one has that (277) vanishes to infinite order at $x = 0$ by Corollary 32.

Proposition 36. *Let $q > 1$. Under the assumptions and notation of Theorem 23, with $\mathscr{W}_{\mu,\lambda}(t)$ as in Definition 16, for $\gamma = 1$ and $a_0 = q > 1$ with $Q = q^{2/\lambda} = q^{\delta/\gamma} = q^\delta$ whereby $q = Q^{1/\delta} = a_0$, one has that*

$$G_0[\mathscr{W}_{\mu,\lambda}](x) = \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \theta\left(Q^2 ; \frac{Q^{2B\alpha/\delta}}{Q^{2+2/\delta}|z_3^{2\delta}(x)|^{2\delta}}\right) \cdot \left| \frac{\delta \mu_Q^3 e^{i\pi\alpha/\beta}}{\sqrt{2\pi}} \frac{1}{(-ix)} \frac{|z_3(x)|^{B\alpha}}{\theta\left(Q ; z_3^\delta(x)\right)} \right|^2 \tag{281}$$

$$= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \theta\left(Q^2 ; \frac{Q^{2B\alpha/\delta}}{Q^{2+2/\delta}|z_3^{2\delta}(x)|^{2\delta}}\right) \left|\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)\right|^2, \tag{282}$$

which is $\mathscr{C}^\infty$ and never vanishes for $x \in \mathbb{R}^*$, in particular for $|x| \in [1, q] = [1, a_0]$.

And for $\gamma = 2$ and $a_0 = q^2 > 1$ with $Q = q^{2/\lambda} = q^{\delta/\gamma} = q^{\delta/2}$ whereby $q^2 = Q^{4/\delta} = a_0$, one has that

$$G_0[\mathscr{W}_{\mu,\lambda}](x) = \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \theta\left(Q^8 ; \frac{Q^{4B\alpha/\delta}}{Q^{6+8/\delta}|z_3(x)|^{4\delta}}\right) \cdot \left| \frac{\delta \mu_Q^3 e^{i\pi\alpha/\beta}}{2\sqrt{2\pi}} \frac{(-1)^j}{(-ix)} \frac{2[z_3(x)]^{B\alpha+j\delta} \theta\left(Q^4 ; z_3^{2\delta}(x)/Q^{1+2j}\right)}{\theta\left(Q ; z_3^\delta(x)\right)\theta\left(Q ; -z_3^\delta(x)\right)} \right|^2 \tag{283}$$

$$= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \theta\left(Q^8 ; \frac{Q^{4B\alpha/\delta}}{Q^{6+8/\delta}|z_3(x)|^{4\delta}}\right) \left|\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)\right|^2, \tag{284}$$

where in (283) $j = 0$ if $B\alpha$ is even and $j = 1$ if $B\alpha$ is odd. Expressions (283)–(284) are $\mathscr{C}^\infty$ and never vanish for $x \in \mathbb{R}^*$, in particular for $|x| \in [1, q^2] = [1, a_0]$.

Proof. First, we handle the $\gamma = 1$ case with $a_0 = q = Q^{1/\delta}$. Under this assumption, along with the assumptions of Theorem 23, one has that (180)–(182) become

$$\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x) = \widehat{C} \frac{1}{(-ix)} \frac{[z_3(x)]^{B\alpha}}{\theta\left(Q ; z_3^\delta(x)\right)}, \quad \text{where } \widehat{C} = \frac{\delta \mu_Q^3 e^{i\pi\alpha/\beta}}{\sqrt{2\pi}}. \tag{285}$$

From (184), one has $z_3(Q^p x) = Q^{p/\gamma} z_3(x) = Q^p z_3(x)$ for $p \in \mathbb{R}$, whence $z_3^\delta(Q^{n/\delta} x) = Q^n z_3^\delta(x)$ for $n \in \mathbb{Z}$ in the theta function expression in (285). One concludes that

$$\begin{aligned} \mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)]\left(Q^{n/\delta} x\right) &= \widehat{C} \frac{1}{\left(-iQ^{n/\delta} x\right)} \frac{\left[z_3\left(Q^{n/\delta} x\right)\right]^{B\alpha}}{\theta\left(Q ; z_3^\delta\left(Q^{n/\delta} x\right)\right)} \\ &= \widehat{C} \frac{1}{\left(-iQ^{n/\delta} x\right)} \frac{Q^{nB\alpha/\delta}[z_3(x)]^{B\alpha}}{\theta\left(Q ; Q^n z_3^\delta(x)\right)} \\ &= \widehat{C} \frac{1}{\left(-iQ^{n/\delta} x\right)} \frac{Q^{nB\alpha/\delta}[z_3(x)]^{B\alpha}}{Q^{n(n+1)/2}\left[z_3^\delta(x)\right]^n \theta\left(Q ; z_3^\delta(x)\right)}, \end{aligned} \tag{286}$$

where equality in (286) follows from (12). From (286) and the fact that $a_0 = q = Q^{1/\delta}$, we have (288) below:

$$G_0[\mathscr{W}_{\mu,\lambda}(t)](x) = \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \sum_{n=-\infty}^{\infty} \left|\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)]\left(Q^{n/\delta} x\right)\right|^2 \tag{287}$$

$$
\begin{aligned}
&= \frac{|\widehat{C}|^2}{\|\mathscr{W}_{\mu,\lambda}\|^2} \sum_{n=-\infty}^{\infty} \left[\frac{Q^{2nB\alpha/\delta}}{Q^{2n/\delta}\left(Q^2\right)^{n(n+1)/2} \left|z_3^{\delta}(x)\right|^{2n}} \left| \frac{1}{(-ix)} \frac{[z_3(x)]^{B\alpha}}{\theta(Q\,;z_3^{\delta}(x))} \right|^2 \right] \\
&= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \left| \frac{\widehat{C}}{(-ix)} \frac{[z_3(x)]^{B\alpha}}{\theta(Q\,;z_3^{\delta}(x))} \right|^2 \\
&\quad \cdot \sum_{n=-\infty}^{\infty} \left[\frac{\left[Q^{2B\alpha/\delta} Q^{-2/\delta} |z_3(x)|^{-2\delta}\right]^n}{\left(Q^2\right)^{n(n+1)/2}} \left[\frac{\left(Q^2\right)^{-2n/2}}{\left(Q^2\right)^{-2n/2}} \right] \right]
\end{aligned} \tag{288}
$$

$$
= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \left| \frac{\widehat{C}}{(-ix)} \frac{[z_3(x)]^{B\alpha}}{\theta(Q\,;z_3^{\delta}(x))} \right|^2 \sum_{n=-\infty}^{\infty} \frac{\left[Q^{2B\alpha/\delta} Q^{-2/\delta} |z_3(x)|^{-2\delta} Q^{-2}\right]^n}{\left(Q^2\right)^{n(n-1)/2}} \tag{289}
$$

$$
= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \left| \frac{\widehat{C}}{(-ix)} \frac{[z_3(x)]^{B\alpha}}{\theta(Q\,;z_3^{\delta}(x))} \right|^2 \theta\left(Q^2\,; Q^{2B\alpha/\delta} Q^{-2/\delta} |z_3(x)|^{-2\delta} Q^{-2}\right), \tag{290}
$$

where the move from (289) to (290) is justified by (10). Now (290) is equal to (281), which by (285), is equal to (282). From its definition in (287), $G_0[\mathscr{W}_{\mu,\lambda}(t)](x)$ is invariant under multiplication of the argument x by $Q^{1/\delta} = (q^{\delta/\gamma})^{1/\delta} = q^{1/\gamma} = q$. This q-invariance can also be checked via properties of the theta functions in expression (281). From (277) in Corollary 34 coupled with (285), one has nonvanishing of $\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)$ for $x \in \mathbb{R}^*$. Since $\theta(Q^2\,;z)$ only vanishes for $z = -Q^{2k}$ for $k \in \mathbb{Z}$ by (10), we have that $\theta(Q^2\,;Q^{2B\alpha/\delta}Q^{-2/\delta}|z_3(x)|^{-2\delta}Q^{-2})$ never vanishes for any $x \in \mathbb{R}^*$ as its argument is positive. From these two nonvanishing results applied to (282) one has that $G_0[\mathscr{W}_{\mu,\lambda}(t)](x)$ is $\mathscr{C}^\infty$ and never vanishes for $x \in \mathbb{R}^*$. The $\gamma = 1$ case is now shown. □

We turn next to the $\gamma = 2$ case, with $a_0 = q^2 = Q^{4/\delta}$. In this setting, from Theorem 23 one has that (180)–(182) becomes

$$
\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x) = \frac{\widehat{C}}{2} \frac{1}{(-ix)} \left(\frac{[z_3(x)]^{B\alpha}}{\theta(Q\,;z_3^{\delta}(x))} + \frac{[-z_3(x)]^{B\alpha}}{\theta(Q;-z_3^{\delta}(x))} \right) \tag{291}
$$

$$
= \frac{\widehat{C}}{2} \frac{1}{(-ix)} (-1)^j\, 2\, z_3^{B\alpha+j\delta}(x) \left[\frac{\theta\left(Q^4\,;\left(z_3^{2\delta}(x)/Q^{1+2j}\right)\right)}{\theta(Q\,;z^{\delta}(x))\,\theta(Q;-z_3^{\delta}(x))} \right], \tag{292}
$$

$$
\text{where} \begin{cases} j = 0, & \text{if } B\alpha \text{ is even}, \\ j = 1, & \text{if } B\alpha \text{ is odd}, \end{cases} \tag{293}
$$

$$
\text{where } \widehat{C} = \frac{\delta\, \mu_Q^3\, e^{i\pi\alpha/\beta}}{\sqrt{2\pi}}, \tag{294}
$$

and where (292) and (293) follow from (275) and (276). From (184), one has $z_3(Q^p x) = Q^{p/\gamma} z_3(x) = Q^{p/2} z_3(x)$ for $p \in \mathbb{R}$, whence $z_3^{\delta}(Q^{4n/\delta}x) = Q^{2n} z_3^{\delta}(x)$ and $z_3^{2\delta}(Q^{4n/\delta}x) = Q^{4n} z_3^{2\delta}(x)$ for $n \in \mathbb{Z}$ in the theta function expressions in (292). From (292), along with the fact that with $a_0 = q^2 = Q^{4/\delta}$, one then has that

$$
\begin{aligned}
&\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)]\left(Q^{4n/\delta}x\right) \\
&\quad = \frac{\widehat{C}}{2} \frac{1}{\left(-iQ^{4n/\delta}x\right)} (-1)^j 2 z_3^{B\alpha+j\delta}\left(Q^{4n/\delta}x\right) \\
&\qquad \cdot \left[\frac{\theta\left(Q^4\,;z_3^{2\delta}\left(Q^{4n/\delta}x\right)/Q^{1+2j}\right)}{\theta\left(Q\,;z^{\delta}\left(Q^{4n/\delta}x\right)\right)\theta\left(Q;-z_3^{\delta}\left(Q^{4n/\delta}x\right)\right)} \right] \\
&\quad = \frac{\widehat{C}}{2} \frac{(-1)^j\, 2 \left[Q^{2n/\delta} z_3(x)\right]^{B\alpha+j\delta}}{\left(-iQ^{4n/\delta}x\right)} \left[\frac{\theta\left(Q^4\,;Q^{4n} z_3^{2\delta}(x)/Q^{1+2j}\right)}{\theta\left(Q\,;Q^{2n} z^{\delta}(x)\right)\theta\left(Q;-Q^{2n} z_3^{\delta}(x)\right)} \right] \\
&\quad = \frac{\widehat{C}}{2} \frac{(-1)^j\, 2 \left[Q^{2n/\delta} z_3(x)\right]^{B\alpha+j\delta}}{\left(-iQ^{4n/\delta}x\right)} \\
&\qquad \cdot \left[\frac{\left(Q^4\right)^{n(n+1)/2} \left[z_3^{2\delta}(x)/Q^{1+2j}\right]^n \theta\left(Q^4\,;z_3^{2\delta}(x)/Q^{1+2j}\right)}{Q^{2n(2n+1)/2} z^{2n\delta}(x)\,\theta\left(Q\,;z^{\delta}(x)\right) Q^{2n(2n+1)/2} (-1)^n z^{2n\delta}(x)\,\theta\left(Q;-z_3^{\delta}(x)\right)} \right] \\
&\quad = \frac{\left[Q^{-4/\delta} Q^{2(B\alpha+j\delta)/\delta} Q^{-1-2j} (-1) z_3^{-2\delta}(x)\right]^n}{Q^{2n^2}}
\end{aligned} \tag{295}
$$

$$
\cdot \left[\frac{\widehat{C}}{2} \frac{1}{(-ix)} (-1)^j\, 2\, z_3^{B\alpha+j\delta}(x) \frac{\theta\left(Q^4\,;z_3^{2\delta}(x)/Q^{1+2j}\right)}{\theta\left(Q\,;z^{\delta}(x)\right)\theta\left(Q;-z_3^{\delta}(x)\right)} \right] \tag{296}
$$

$$
= \frac{\left[Q^{-4/\delta} Q^{2(B\alpha+j\delta)/\delta} Q^{-1-2j} (-1) z_3^{-2\delta}(x) Q^{-2}\right]^n}{\left(Q^4\right)^{n(n-1)/2}} \left[\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)\right], \tag{297}
$$

where j is as in (293); (12) was used to move from (295) to the subsequent line; all terms involving a power of n have been factored out in (296); and in moving to (297), the expressions involving powers of n have been multiplied by Q^{-2n}/Q^{-2n} while the bracketed expression in (296) was recognized as the Fourier transform of $\mathscr{W}_{\mu,\lambda}(t)$ via (292). One then uses (297) to compute the diagonal term $G_0[\mathscr{W}_{\mu,\lambda}(t)](x)$ as follows:

$$
\begin{aligned}
G_0[\mathscr{W}_{\mu,\lambda}(t)](x) &= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \sum_{n=-\infty}^{\infty} \left| \mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)]\left(Q^{4n/\delta}x\right) \right|^2 \\
&= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \left| \mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x) \right|^2
\end{aligned} \tag{298}
$$

$$
\cdot \sum_{n=-\infty}^{\infty} \left[\frac{\left[Q^{-8/\delta} Q^{4(B\alpha+j\delta)/\delta} Q^{-2-4j} |z_3(x)|^{-4\delta} Q^{-4}\right]^n}{\left(Q^8\right)^{n(n-1)/2}} \right] \tag{299}
$$

$$
= \frac{1}{\|\mathscr{W}_{\mu,\lambda}\|^2} \left| \mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x) \right|^2 \theta\left(Q^8\,; \frac{Q^{4B\alpha/\delta}}{Q^{6+8/\delta} |z_3(x)|^{4\delta}} \right), \tag{300}
$$

where (297) is used to move from (298) to (299) (after taking absolute value and squaring) and (10) is used to move from (299) to (300). Finally, (300) is seen to be equivalent to (284), which is in turn equivalent to (283). From its definition in (298), $G_0[\mathscr{W}_{\mu,\lambda}(t)](x)$ is invariant under multiplication of the argument x by $Q^{4/\delta} = (q^{4\delta/\gamma})^{1/\delta} = q^{4/\gamma} = q^{4/2} = q^2$. This q^2-invariance can also be checked via properties of the theta functions in expression (300). From (277) in Corollary 34 coupled with (291), one has nonvanishing of $\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)$ for $x \in \mathbb{R}^*$. Since $\theta(Q^8\,; z)$ only vanishes for $z = -Q^{8k}$ for $k \in \mathbb{Z}$ by (10), we have that $\theta(Q^8\,; Q^{4B\alpha/\delta}Q^{-6-(8/\delta)}|z_3(x)|^{-4\delta})$ never vanishes for any $x \in \mathbb{R}^*$ as its argument is positive. From these two nonvanishing results applied to (300), one has that $G_0[\mathscr{W}_{\mu,\lambda}(t)](x)$ is $\mathscr{C}^\infty$ and never vanishes for $x \in \mathbb{R}^*$. The $\gamma = 2$ case is now shown, and the proposition is proven.

At this point, at least for low values of $\gamma = 1, 2$, we are prepared to explicitly make the connection to wavelet frame theory that follows from knowledge of the Fourier transform of $\mathscr{W}_{\mu,\lambda}(t)$ in terms of the theta function.

Theorem 37. *Let $q > 1$. Under the assumptions and notation of Theorem 23, in particular $\delta = 0 \bmod \beta$, and with $\mathscr{W}_{\mu,\lambda}(t)$ as in Definition 16, one has, for $\gamma = 1$ with $a_0 = q$, and also for $\gamma = 2$ with $a_0 = q^2$, that for $b_0 > 0$ sufficiently small*

$$\mathscr{W}_{\mu,\lambda}(t) \text{ is a mother wavelet for a frame } S(\mathscr{W}_{\mu,\lambda}\,; a_0, b_0) \text{ generating } \mathscr{L}^2(\mathbb{R}). \tag{301}$$

Proof. The theorem will follow by establishing (264) above. Now, $G_0[\mathscr{W}_{\mu,\lambda}](x) \neq 0$ for $1 \le |x| \le a_0$ by Proposition 36. Next, from Theorem 23, we have that $\mathscr{W}_{\mu,\lambda}(t)$ is a Schwartz wavelet with all moments vanishing. Since all moments vanish, $\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)$ is flat at $x = 0$. Since $\mathscr{W}_{\mu,\lambda}(t)$ is Schwartz, $\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x)$ is also Schwartz and therefore decays faster than $1/|x|^p$ for any $p \in \mathbb{N}$ for x near $\pm\infty$. Hence, for choice of C sufficiently large, one has $\mathscr{F}[\mathscr{W}_{\mu,\lambda}(t)](x) \le C|x|/(1 + x^2)^{3/2}$. Thus, (264) is satisfied, and the theorem is demonstrated. ☐

5. Canonical Extensions

From Theorem 3.2 in [2], a given $f_{\mu,\lambda}(t)$ on $[0,\infty)$ has, in general, infinitely many Schwartz wavelet extensions $F_{\mu,\lambda}(t)$ to all of $\mathbb{R}$ with $F_{\mu,\lambda}(t)$ satisfying the same MADE as $f_{\mu,\lambda}(t)$. In this section, we shall demonstrate, in the setting of Theorem 23 for low values of $M = \delta = 1, 2, 3$, that there is a natural uniquely determined extension $F_{\mu,\lambda}(t)$ of $f_{\mu,\lambda}(t)$ to $\mathbb{R}$, namely, the canonical extension $\mathscr{W}_{\mu,\lambda}(t)$ of $f_{\mu,\lambda}(t)$ given as follows. We remark that for these values 1, 2, 3 one has $\delta \neq 0 \bmod 4$ as in Theorem 23 automatically.

Definition 38. Under the assumptions and notation of Theorem 23 (in particular $\delta = 0 \bmod \beta$), with $M = \delta = 1, 2, 3$ (and $\delta = 0 \bmod \beta$), the canonical extension of $f_{\mu,\lambda}(t)$ from $[0,\infty)$ to all of $\mathbb{R}$ is defined to be $\mathscr{W}_{\mu,\lambda}(t)$, where $\mathscr{W}_{\mu,\lambda}(t)$ is the function naturally generated by $f_{\mu,\lambda}(t)$ as given by Definition 16, with $\mathscr{W}_{\mu,\lambda}(t)|_{[0,\infty)} = f_{\mu,\lambda}(t)$.

We clarify the above definition by emphasizing that canonical extensions of $f_{\mu,\lambda}(t)$ as the μ, λ vary (as given in Definition 38) form a strict subset of the set of functions naturally generated by $f_{\mu,\lambda}(t)$ (as in Definition 16) . This follows from the fact that, for $\delta > 3$, in a large number of cases one has that $\mathscr{W}_{\mu,\lambda}(t) \neq f_{\mu,\lambda}(t)$ for $t \ge 0$ and is therefore not an extension (see Propositions 42 and 43 below). We next show that for $\delta = 1, 2, 3, \mathscr{W}_{\mu,\lambda}(t)$ is indeed an extension of $f_{\mu,\lambda}(t)$.

Proposition 39. *Under the assumptions and notation of Theorem 23, for $\delta = 1, 2, 3$, with $\delta = 0 \bmod \beta$ one has that $\mathscr{W}_{\mu,\lambda}(t)|_{[0,\infty)} = f_{\mu,\lambda}(t)$ for $t \ge 0$, where $f_{\mu,\lambda}(t)$ is given by (1).*

Proof. We handle each value of δ separately. First, let $\delta = 1$. In this setting, one has $\omega = e^{2\pi i/M} = e^{2\pi i/\delta} = e^{2\pi i/1} = 1$. From (131) and (132), one has that

$$\begin{aligned}\mathscr{W}_{\mu,\lambda}(t) &= \sum_{\ell=0}^{\delta-1} \left[\omega^{\ell+1}\right]^{B\alpha} \tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \\ &= \sum_{\ell=0}^{0} [1]^{B\alpha} \tilde{f}_{\mu,\lambda}\left([1]^{D\gamma} t\right) = \tilde{f}_{\mu,\lambda}(t),\end{aligned} \tag{302}$$

and hence, $\mathscr{W}_{\mu,\lambda}(t)|_{[0,\infty)} = f_{\mu,\lambda}(t)$ on $[0,\infty)$ holds for $\delta = 1$ because $\tilde{f}_{\mu,\lambda}(t) = f_{\mu,\lambda}(t)$ for $t \ge 0$. This last equality follows from (125) in conjunction with (1) after noting that $Q = q^{\delta/\gamma}$ and so $[Q^{k/M}]^{D\gamma} = [(q^{\delta/\gamma})^{k/d}]^{1\gamma} = q^k$. We remark that, from Case $M = 1$ in Section 6.1 below, the $\delta = 1$ Case under consideration here consists of precisely the $f_{\mu,\lambda}(t)$ that are flat at the origin and the canonical extension is $f_{\mu,\lambda}(t)$ extended to be 0 on the negative real axis, equivalently the canonical extension is $\tilde{f}_{\mu,\lambda}(t) = \mathscr{W}_{\mu,\lambda}(t)$. ☐

Next, let $\delta = 2$. Since $\gamma/\delta = \gamma/2$ is in reduced form, one concludes that $\gamma = 2k + 1$ is odd. Now, $\omega = e^{2\pi i/M} = e^{2\pi i/\delta} = e^{2\pi i/2} = -1$. From (131)-(132), one has that

$$\begin{aligned}\mathscr{W}_{\mu,\lambda}(t) &= \sum_{\ell=0}^{\delta-1} \left[\omega^{\ell+1}\right]^{B\alpha} \tilde{f}_{\mu,\lambda}\left(\left[\omega^{\ell+1}\right]^{D\gamma} t\right) \\ &= \sum_{\ell=0}^{2-1} \left[(-1)^{\ell+1}\right]^{B\alpha} \tilde{f}_{\mu,\lambda}\left(\left[(-1)^{\ell+1}\right]^{1(2k+1)} t\right) \\ &= [-1]^{B\alpha}\tilde{f}_{\mu,\lambda}((-1)t) + [1]^{B\alpha}\tilde{f}_{\mu,\lambda}(1t)\end{aligned} \tag{303}$$

$$= \begin{cases} [1]^{B\alpha} f_{\mu,\lambda}(t), & \text{for } t \ge 0, \\ [1]^{B\alpha}(-1) f_{\mu,\lambda}((-1)t), & \text{for } t < 0, \end{cases} \tag{304}$$

where one moves from (303) to (304) via (125) and (128). From (304), one sees that $\mathscr{W}_{\mu,\lambda}(t)|_{[0,\infty)} = f_{\mu,\lambda}(t)$ when $\delta = 2$. Also note from (304) that the canonical extension $\mathscr{W}_{\mu,\lambda}(t)$ is an even function if $B\alpha$ is odd, and it is an odd function if $B\alpha$ is even when $\delta = 2$.

Finally, let $\delta = 3$. Since $\gamma/\delta = \gamma/3$ is in reduced form, one concludes that $\gamma = 3p + j$ for $p \in \mathbb{N}_0$ with $j = 1, 2$. In this setting, $\omega = e^{2\pi i/M} = e^{2\pi i/\delta} = e^{2\pi i/3}$ which gives that

$$[\omega^{\ell+1}]^{D\gamma} = [\omega^{\ell+1}]^{1(3p+j)} = \left[(e^{2\pi i/3})^{(3p+j)}\right]^{\ell+1} \quad (305)$$

$$= [e^{2\pi ij/3}]^{\ell+1} = \begin{cases} e^{2\pi ij/3}, & \text{if } \ell = 0, \\ e^{4\pi ij/3}, & \text{if } \ell = 1. \\ 1, & \text{if } \ell = 2. \end{cases} \quad (306)$$

For each case that $j = 1, 2$ one has that $\mathcal{R}(e^{2\pi ij/3}) < 0$ and $\mathcal{R}(e^{4\pi ij/3}) < 0$. From (306), $[\omega^{\ell+1}]^{D\gamma}$ has negative real part for $\ell = 0, 1$ and $[\omega^{\ell+1}]^{D\gamma} = 1$ for $\ell = 2$. Then for $t \geq 0$ and for $\ell = 0, 1$, (128) gives that $\tilde{f}_{\mu,\lambda}([\omega^{\ell+1}]^{D\gamma} t) = 0$. Combining these results with (131) and (132) and (125) gives for $t \geq 0$ that

$$\begin{aligned} \mathcal{W}_{\mu,\lambda}(t) &= \sum_{\ell=0}^{\delta-1} [\omega^{\ell+1}]^{B\alpha} \tilde{f}_{\mu,\lambda}\left([\omega^{\ell+1}]^{D\gamma} t\right) \\ &= \sum_{\ell=0}^{2} [\omega^{\ell+1}]^{B\alpha} \tilde{f}_{\mu,\lambda}\left([\omega^{\ell+1}]^{D\gamma} t\right) \end{aligned} \quad (307)$$

$$= 0 + 0 + [\omega^{2+1}]^{B\alpha} \tilde{f}_{\mu,\lambda}\left([\omega^{2+1}]^{D\gamma} t\right) = \tilde{f}_{\mu,\lambda}(t) = f_{\mu,\lambda}(t). \quad (308)$$

Hence, in each of the cases $\delta = 1, 2, 3$ one has $\mathcal{W}_{\mu,\lambda}(t)|_{[0,\infty)} = f_{\mu,\lambda}(t)$, and the proposition is proven.

In the next theorem, we assume all of the hypotheses of both Theorems 13 and 23 to determine a family of $f_{\mu,\lambda}(t)$ with canonical extensions having optimal properties.

Theorem 40. *Let $q > 1$. Assume that all the hypotheses and the notation of Theorems 13 and 23 hold. Then for*

$$(\gamma, \delta) \text{ satisfying } \gamma \in \{1, 2\} \text{ and } \delta \in \{1, 2, 3\}, \quad (309)$$

equivalently, for

$$(\mu, \lambda) \in (\mathbb{Z}, 1) \text{ or } (2\mathbb{Z}+1, 2) \text{ or } (2\mathbb{Z}+1, 4) \text{ or } \cdot (2\mathbb{Z}+1+2j/3, 2/3) \text{ or } (2\mathbb{Z}+1+2j/3, 4/3) \text{ for } j = 0, 1, 2, \quad (310)$$

the associated $f_{\mu,\lambda}(t)$ have canonical extensions $\mathcal{W}_{\mu,\lambda}(t)$ which

(1) are Schwartz wavelets on $\mathbb{R}$

(2) are mother wavelets for a frame $S(\mathcal{W}_{\mu,\lambda}; a_0, b_0)$ generating $\mathcal{L}^2(\mathbb{R})$, as in (301), where b_0 is sufficiently small, and $a_0 = q > 1$ when $\gamma = 1$ and $a_0 = q^2 > 1$ when $\gamma = 2$

(3) have all moments vanishing

(4) have Fourier transforms given by (180)-(182) which relate the canonical extension of $f_{\mu,\lambda}(t)$ to the Jacobi theta function

(5) satisfy the MADE (185)

(6) have nonvanishing diagonal frame terms $G_0[\mathcal{W}_{\mu,\lambda}](x)$ on $[1, a_0]$: given by (281)-(282) for $\gamma = 1$ and $a_0 = q$, and given by (283)-(284) for $\gamma = 2$ and $a_0 = q^2$, each expressible in terms of the Jacobi theta function

Proof. Since $\delta \in \{1, 2, 3\}$ one has $\mathcal{W}_{\mu,\lambda}(t)$ is a canonical extension of $f_{\mu,\lambda}(t)$ to $\mathbb{R}$ by Proposition 39 and Definition 38. Properties (1), (3), (4), and (5) follow from Theorem 23. Property (2) follows from Theorem 37. Since $\gamma \in \{1, 2\}$, Property (6) follows from Proposition 36. It remains to show the equivalence of (297) with (310). □

We first show (297) ⇒ (310). Assuming (297), one has $\lambda/2 = \gamma/\delta \in \{1/1, 2/1, 1/2, 1/3, 2/3\}$ where the case that $\gamma/\delta = 2/2$ is ruled out for not being in reduced form. From this, one concludes $\lambda \in \{1, 2, 4, 2/3, 4/3\}$. Since $\delta = 0 \mod \beta$, one concludes that $\beta = 1$ when $\delta = 1$; $\beta = 1, 2$ when $\delta = 2$; and $\beta = 1, 3$ when $\delta = 3$.

When $\beta = 1$, which happens for each value of λ, one has

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{1} \text{ from which } \mu = 2\alpha - 1 \text{ is odd.} \quad (311)$$

Thus, the $\beta = 1$ case gives that (μ, λ) falls in one of $(2\mathbb{Z}+1, 1)$, $(2\mathbb{Z}+1, 2)$, $(2\mathbb{Z}+1, 4)$, $(2\mathbb{Z}+1, 2/3)$, or $(2\mathbb{Z}+1, 4/3)$.

When $\beta = 2$, which happens only when $\delta = 2$ (for $\delta \in \{1, 2, 3\}$), equivalently, when $\lambda/2 = \gamma/\delta = 1/2$, equivalently, when $\lambda = 1$, one has

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{2k+1}{2}, \quad \text{where } \alpha \text{ must be odd, as } \frac{\alpha}{2} \text{ is in a reduced form.} \quad (312)$$

Hence, $\mu = 2k$ is even when $\beta = 2$, and in this case, (μ, λ) falls in $(2\mathbb{Z}, 1)$.

When $\beta = 3$, which happens only when $\delta = 3$ (for $\delta \in \{1, 2, 3\}$), equivalently, when $\lambda/2 = \gamma/\delta = \gamma/3$, equivalently, when $\lambda = 2/3, 4/3$, one has

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{3k+j}{3}, \quad \text{with } j = 1, 2 \text{ since } \frac{\alpha}{3} \text{ is in a reduced form.} \quad (313)$$

Hence, $\mu = 2k - 1 + 2j/3$ for $j = 1, 2$ when $\beta = 3$, and in this case, (μ, λ) falls in $(2\mathbb{Z}+1+2j/3, 2/3)$ or $(2\mathbb{Z}+1+2j/3, 4/3)$.

All of the above cases $\beta = 1, 2, 3$ combine to require that (μ, λ) satisfies (310).

Conversely, we show (310) $\Rightarrow$ (309). So assume (310). Within this setting, examine the case that $\mu = 2k - 1$ is odd, with $\lambda \in \{1, 2, 4, 2/3, 4/3\}$. Then,

$$\frac{\mu+1}{2} = \frac{2k-1+1}{2} = k = \frac{k}{1} = \frac{\alpha}{\beta} \quad \text{in a reduced form,} \tag{314}$$

from which we deduce that $\beta = 1$ and $\gamma/\delta = \lambda/2 \in \{1/2, 1, 2, 1/3, 2/3\}$. This implies $\delta = 1, 2, \text{or } 3$, none of which is ruled out as each such δ is divisible by $\beta = 1$. Furthermore, from the possible $\lambda/2$, one sees that $\gamma \in \{1, 2\}$.

Next, examine the case that $\mu = 2k$ is even with $\lambda = 1$. Then, $\mu + 1 = 2k + 1$. Hence,

$$\frac{\mu+1}{2} = \frac{2k+1}{2} = \frac{\alpha}{\beta} \quad \text{which is in reduced form.} \tag{315}$$

One concludes that $\beta = 2$ and $\lambda/2 = \gamma/\delta \in \{1/2\}$, in a reduced form. Thus, $\delta = 2$ (which is divisible by $\beta = 2$), and $\gamma = 1$.

Finally, examine the case that $\mu = 2k - 1 + 2j/3$ with $j = 1, 2$ and $\lambda \in \{2/3, 4/3\}$. Then,

$$\frac{\mu+1}{2} = \frac{3k+j}{3} = \frac{\alpha}{\beta} \quad \text{which is in reduced form as } j = 1, 2. \tag{316}$$

One concludes that $\beta = 3$ which does divide δ for $\lambda/2 = \gamma/\delta \in \{1/3, 2/3\}$, all in a reduced form. Thus, $\delta = 3$ and $\gamma = 1, 2$.

All cases combine to give $\gamma \in \{1, 2\}$ and $\delta \in \{1, 2, 3\}$, which is (309). Thus, (309) $\Longleftrightarrow$ (310), and the theorem is now proven.

Selected examples of canonical extensions are now provided.

[Example $\mu = -1$ odd and $\lambda = 2$] In this case,

$$f_{-1,2}(t) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \exp\left(-q^k t\right)}{q^{k(k+1)/2}}, \quad \text{for } t \geq 0. \tag{317}$$

One sees that $(\mu + 1)/2 = 0/2 = 0/1 = \alpha/\beta$ and $\lambda/2 = 2/2 = 1/1 = \gamma/\delta$. Thus, $\alpha = 0$ and $\gamma = 1$, while $\beta = \delta = 1$, consistent with Examples $[M = 1]$ in Section 6.1 and $[M = \delta = 1, \mu = 2N + 1, \text{and } \lambda = 2n]$ in Section 6.2 below, where the current case that $\mu = -1$ and $\lambda = 2$ is seen to be flat at the origin. One extends $f_{-1,2}(t)$ to be identically 0 for $t < 0$. This yields $\tilde{f}_{-1,2}(t)$, which equals the canonical extension $\mathscr{W}_{-1,2}(t)$ by (302) as $\delta = 1$. This particular canonical extension was first introduced in [3] as the function $K(t)$. From Theorem 40, one has that $K(t)$ satisfies properties (1) through (6), including that $K(t)$ is a Schwartz wavelet with all moments vanishing, satisfying the MADE $K'(t) = K(qt)$, generating a frame for $\mathscr{L}^2(\mathbb{R})$, and having Fourier transform $i\mu_q^3/[\sqrt{2\pi} x\theta(q; ix)]$, as was previously proven in [3] but is now seen as a special case of Theorem 40.

[Example $\mu = 0$ even and $\lambda = 1$] In this case,

$$f_{0,1}(t) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \exp\left(-q^k t\right)}{q^{k^2}}, \quad \text{for } t \geq 0. \tag{318}$$

One sees that $(\mu + 1)/2 = 1/2 = \alpha/\beta$ and $\lambda/2 = 1/2 = \gamma/\delta$. Thus, $\alpha = \gamma = 1$, while $\beta = \delta = 2$, consistent with Examples $[M = \delta = p, \beta = p]$ (with the prime p taken to be 2) in Section 6.1 and $[M = \delta = 2, \beta = 2]$ in Section 6.2 below. Also, we record that $B = M/\beta = \delta/\beta = 2/2 = 1$. Since $\delta = 2$, $f_{0,1}(t)$ has canonical extension $\mathscr{W}_{0,1}$ given by (304). That is,

$$\begin{aligned}
\mathscr{W}_{0,1}(t) &= \begin{cases} [1]^{B\alpha} f_{0,1}(t), & \text{for } t \geq 0 \\ [-1]^{B\alpha}(-1) f_{0,1}((-1)t), & \text{for } t < 0 \end{cases} \\
&= \begin{cases} [1]^{1\cdot 1} f_{0,1}(t), & \text{for } t \geq 0 \\ [-1]^{1\cdot 1}(-1) f_{0,1}((-1)t), & \text{for } t < 0 \end{cases} \\
&= f_{0,1}(|t|) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \exp\left(-q^k |t|\right)}{q^{k^2}},
\end{aligned} \tag{319}$$

which is the extension of $f_{0,1}$ to be an even function. This canonical extension $\mathscr{W}_{0,1}(t)$ was first introduced in [4] and denoted there by $f_0(t)$. From Theorem 40, one has that $f_0(t)$ satisfies properties (1) through (6), including that $f_0(t)$ is a Schwartz wavelet with all moments vanishing, satisfying the MADE $f_0''(t) = -qf_0(qt)$, generating a frame for $\mathscr{L}^2(\mathbb{R})$, and having Fourier transform $2(\mu_{q^2})^3/[\sqrt{2\pi}\theta(q^2; x^2)]$, as was previously proven in [4] but is now seen as a special case of Theorem 40. If one normalizes $f_0(t)$ by $f_0(0)$, one obtains ${}_q\mathrm{Cos}(t) \equiv f_0(t)/f_0(0)$, which converges uniformly [4] to cos (t) on compact subsets of $\mathbb{R}$ as $q \longrightarrow 1^+$, as is illustrated in Figure 2(a). Furthermore, ${}_q\mathrm{Cos}(t)$ satisfies properties (1) through (6), including satisfying the same MADE above (by linearity) and having Fourier transform $2(\mu_{q^2})^3/[\sqrt{2\pi} f_0(0)\theta(q^2; x^2)]$. See [4] and [6] for further details.

[Example $\mu = 1$ odd and $\lambda = 1$] In this case,

$$f_{1,1}(t) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \exp\left(-q^k t\right)}{q^{k(k-1)}}, \quad \text{for } t \geq 0. \tag{320}$$

One sees that $(\mu + 1)/2 = (1 + 1)/2 = 1/1 = \alpha/\beta$ and $\lambda/2 = 1/2 = \gamma/\delta$. Thus, $\alpha = \beta = \gamma = 1$, while $\delta = 2$, consistent with Examples $[M = \delta = p, \beta = 1]$ in Section 6.1 (with the prime p taken to be 2) and $[M = \delta = 2, \beta = 1]$ in Section 6.2 below. Also, we record that $B = M/\beta = \delta/\beta = 2/1 = 2$. Since $\delta = 2$, $f_{1,1}(t)$ has canonical extension $\mathscr{W}_{1,1}$ given by (290). That is,

$$\mathscr{W}_{1,1}(t)=\begin{cases}[1]^{B\alpha}f_{1,1}(t), & \text{for } t\geq 0\\ [-1]^{B\alpha}(-1)f_{1,1}((-1)t), & \text{for } t<0\end{cases}$$
$$=\begin{cases}[1]^{2\cdot 1}f_{1,1}(t), & \text{for } t\geq 0\\ [-1]^{2\cdot 1}(-1)f_{1,1}((-1)t), & \text{for } t<0\end{cases} \quad (321)$$
$$=\begin{cases}\sum_{k=-\infty}^{\infty}\frac{(-1)^k\exp\left(-q^kt\right)}{q^{k(k-1)}}, & \text{for } t\geq 0,\\ (-1)\sum_{k=-\infty}^{\infty}\frac{(-1)^k\exp\left(-q^k|t|\right)}{q^{k(k-1)}}, & \text{for } t<0,\end{cases}$$

which is the extension of $f_{1,1}$ to be an odd function. This canonical extension $\mathscr{W}_{1,1}(t)$ was first introduced in [4] and denoted there by $f_1(t)$. From Theorem 40, one has that $f_1(t)$ satisfies properties (1) through (6), including that $f_1(t)$ is a Schwartz wavelet with all moments vanishing, satisfying the MADE $f_1'{}'(t)=-q^2f_1(qt)$, generating a frame for $\mathscr{L}^2(\mathbb{R})$, and having Fourier transform $2(\mu_{q^2})^3(-ix)/[\sqrt{2\pi}\theta(q^2\,;x^2)]$, as was previously proven in [4] but is now seen as a special case of Theorem 40. If one normalizes $f_1(t)$ by $f_0(0)$, one obtains ${}_q\mathrm{Sin}(t)\equiv f_1(t)/f_0(0)$, which converges uniformly to $\sin(t)$ [4] on compact subsets of $\mathbb{R}$ as $q\longrightarrow 1^+$, as is illustrated in Figure 2(b). Furthermore, ${}_q\mathrm{Sin}(t)$ satisfies properties (1) through (6), including satisfying the same MADE above (by linearity) and having Fourier transform $2(\mu_{q^2})^3(-ix)/[\sqrt{2\pi}f_0(0)\theta(q^2\,;x^2)]$. Also, ${}_q\mathrm{Sin}'(t)=q\cdot{}_q\mathrm{Cos}(qt)$ while ${}_q\mathrm{Cos}'(t)=-{}_q\mathrm{Sin}(t)$. See [4] and [6] for further details.

While the above examples are consistent with earlier examples in our previous work, the examples for

$$(\mu,\lambda)\in(2\mathbb{Z}+1,4)\text{ or }(2\mathbb{Z}+1+,)\text{ or }\left(2\mathbb{Z}+1+\frac{2j}{3},\right),\quad\text{for } j=0,1,2 \quad (322)$$

are all not previously seen. To illustrate an example from (322), we choose $(\mu,\lambda)\in(2\mathbb{Z}+1+2j/3,2/3)$ for $j=0$ and develop it here.

[Example $\mu=1$ odd and $\lambda=2/3$] In this case,

$$f_{1,2/3}(t)=\sum_{k=-\infty}^{\infty}\frac{(-1)^k\exp\left(-q^kt\right)}{q^{k(k-1)/[2/3]}}\text{ for } t\geq 0. \quad (323)$$

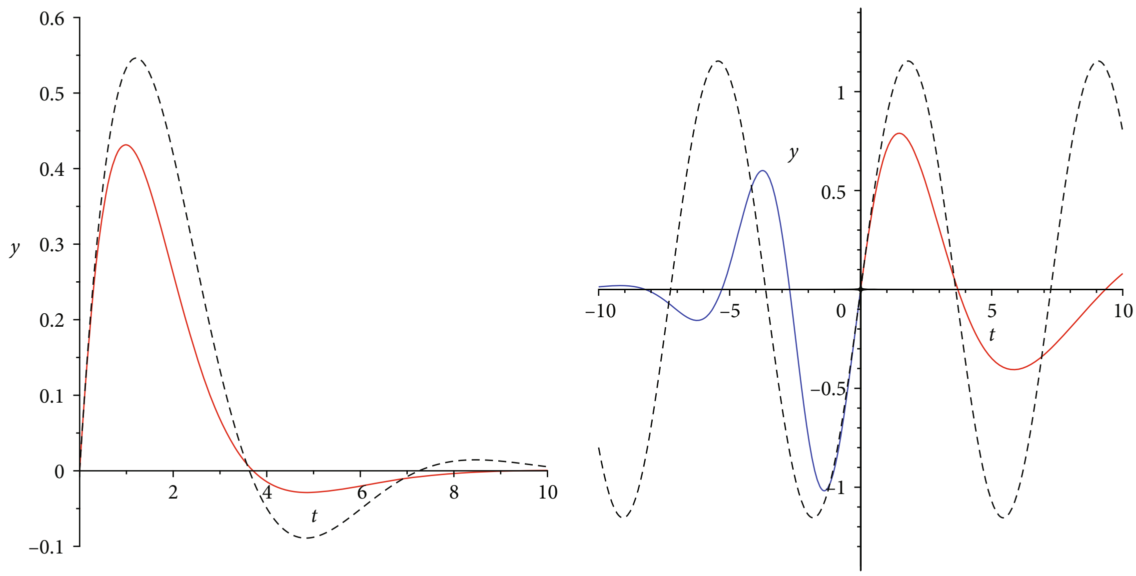
One sees that $(\mu+1)/2=(1+1)/2=1/1=\alpha/\beta$ and $\lambda/2=1/3=\gamma/\delta$. Thus, $\alpha=\beta=\gamma=1$, while $\delta=3$, and $\omega=e^{2\pi i/\delta}=e^{2\pi i/3}$, consistent with Examples $[M=\delta=p,\beta=1]$ (with the prime p taken to be 3) in Section 6.1 and $[M=\delta=3,\beta=1]$ in Section 6.2 below. Also, we record that $B=M/\beta=\delta/\beta=3/1=3$. Since $\delta=3$, $f_{1,2/3}(t)$ has canonical extension $\mathscr{W}_{1,2/3}$ given by (307). That is,

$$\mathscr{W}_{1,2/3}(t)=\sum_{\ell=0}^{\delta-1}\left[\omega^{\ell+1}\right]^{B\cdot\alpha}\tilde{f}_{1,2/3}\left(\left[\omega^{\ell+1}\right]^{D\gamma}t\right)=\sum_{\ell=0}^{2}\left[e^{2\pi i(\ell+1)/3}\right]^{3\cdot 1}\tilde{f}_{1,2/3}\left(\left[e^{2\pi i(\ell+1)/3}\right]^{1\cdot 1}t\right)$$
$$=\sum_{\ell=0}^{2}\tilde{f}_{1,2/3}\left(\left[e^{2\pi i(\ell+1)/3}\right]t\right)=\tilde{f}_{1,2/3}\left(\left[e^{2\pi i/3}\right]t\right)+\tilde{f}_{1,2/3}\left(\left[e^{4\pi i/3}\right]t\right)+\tilde{f}_{1,2/3}(t) \quad (324)$$
$$=\begin{cases}0+0+f_{1,2/3}(t), & \text{for } t\geq 0\\ (-1)f_{1,2/3}\left(\left[e^{2\pi i/3}\right]t\right)+(-1)f_{1,2/3}\left(\left[e^{4\pi i/3}\right]t\right)+0, & \text{for } t<0\end{cases}$$
$$=\begin{cases}\sum_{k=-\infty}^{\infty}(-1)^k\frac{\exp\left(-q^kt\right)}{q^{k(k-1)/[2/3]}}, & \text{for } t\geq 0,\\ (-1)\sum_{k=-\infty}^{\infty}(-1)^k\frac{\exp\left(-q^k\left[e^{2\pi i/3}\right]t\right)}{q^{k(k-1)/[2/3]}}+(-1)\sum_{k=-\infty}^{\infty}(-1)^k\frac{\exp\left(-q^k\left[e^{4\pi i/3}\right]t\right)}{q^{k(k-1)/[2/3]}}, & \text{for } t<0,\end{cases} \quad (325)$$

where (324) follows from Definition 15 and (325) follows from (323) and (9). From Theorem 40, one has that $\mathscr{W}_{1,2/3}(t)$ satisfies properties (1) through (6), including that $\mathscr{W}_{1,2/3}(t)$ is a Schwartz wavelet with all moments vanishing, satisfying the MADE $\mathscr{W}^{(3)}_{1,2/3}(t)=q^3\mathscr{W}_{1,2/3}(qt)$, generating a frame for $\mathscr{L}^2(\mathbb{R})$, and having Fourier transform $-3(\mu_{q^3})^3x^2/[\sqrt{2\pi}\,\theta(q^3;-ix^3)]$.

One computes

$$\mathscr{W}_{1,2/3}(0)=f_{1,2/3}(0)=\sum_{k=-\infty}^{\infty}(-1)^k\frac{1}{q^{k(k-1)/[2/3]}}$$
$$=\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(q^3)^{k(k-1)/2}}=\theta\left(q^3;-1\right)=0, \quad (326)$$



(a) (b)

FIGURE 3: (a) $y = 2\exp(-t/2)\sin(\sqrt{3}t/2)/\sqrt{3}$ (dashed) approximated by $f_{1,2/3}(t)/f'_{1,2/3}(0)$ for $q = 1.3$ (solid red). (b) Scaled $y = 2\sin(\sqrt{3}t/2)/\sqrt{3}$ (dashed) approximated by the scaled canonical extension $\exp(t/2)\mathscr{W}_{1,2/3}(t)/\mathscr{W}'_{1,2/3}(0)$ (solid blue/red) for $q = 1.3$ and t near 0. Each is scaled by $\exp(t/2)$.

while the first derivative at the origin is

$$\mathscr{W}'_{1,2/3}(0) = f'_{1,2/3}(0) = -f_{5/3,2/3}(0) = -\sum_{k=-\infty}^{\infty}(-1)^k\frac{1}{q^{k(k-1-2/3)/[2/3]}} \tag{327}$$

$$= -\sum_{k=-\infty}^{\infty}\frac{(-q)^k}{(q^3)^{k(k-1)/2}} = -\theta(q^3;-q) \neq 0, \tag{328}$$

where the penultimate equality in each of (326) and (328) follows from (10), the second equality in (327) follows from the fact that $f'_{\mu,\lambda}(t) = -f_{\mu+\lambda,\lambda}(t)$ (as is proven in equation (11) of [2]), and the inequality giving nonvanishing in (328) follows from (13). While we are here, we also compute the second derivative at the origin by

$$\begin{aligned}\mathscr{W}'_{1,2/3}{}'(0) = f'_{1,2/3}{}'(0) = f_{7/3,2/3}(0) &= \sum_{k=-\infty}^{\infty}(-1)^k\frac{1}{q^{k(k-1-4/3)/[2/3]}} \\ &= \sum_{k=-\infty}^{\infty}\frac{(-q^2)^k}{(q^3)^{k(k-1)/2}} = \theta(q^3;-q^2) \neq 0,\end{aligned} \tag{329}$$

Normalize $\mathscr{W}_{1,2/3}(t)$ by $\mathscr{W}'_{1,2/3}(0) = -\theta(q^3;-q)$ to obtain $g(q;t) \equiv \mathscr{W}_{1,2/3}(t)/\mathscr{W}'_{1,2/3}(0)$, where $g(q;t)$ satisfies (1) through (6) of Theorem 40, while having Fourier transform $3(\mu_{q^3})^3x^2/[\sqrt{2\pi}\,\theta(q^3;-q)\theta(q^3;-ix^3)]$ and satisfying the same MADE as $\mathscr{W}_{1,2/3}(t)$, namely,

$$g^{(3)}(q;t) = q^3 g(q;qt), \tag{330}$$

with initial conditions

$$\begin{aligned}&g(q;0) = 0,\ g'(q;0) = -\theta(q^3;-q)/[-\theta(q^3;-q)] = 1 \\ &\text{and } g''(q;0) = \theta(q^3;-q^2)/[-\theta(q^3;-q)] = -q.\end{aligned} \tag{331}$$

This second derivative reducing to $-q$ in (331) follows from Lemma 41 below. For small $q > 1$, as $q \longrightarrow 1^+$, (330) and (331) can be considered to be a perturbation of the ODE initial value problem (332) and (333), where

$$f^{(3)}(t) = f(t), \tag{332}$$

with initial conditions

$$f(0) = 0, f'(0) = 1, f''(0) = -1, \tag{333}$$

which is solved by $f(t) = 2\exp(-t/2)\sin(\sqrt{3}t/2)/\sqrt{3}$. Figure 3 provides graphical evidence that $\mathscr{W}_{1,2/3}(t)/\mathscr{W}_{1,2/3}(0)$ converges to $f(t) = 2\exp(-t/2)\sin(\sqrt{3}t/2)/\sqrt{3}$ near $t = 0$. We have scaled Figure 3(b) by $\exp(t/2)$ to better visualize the graph on the negative t axis.

Lemma 41. *For $q > 1$ the Jacobi theta function (10) satisfies*

$$\frac{\theta(q^3;q^2)}{\theta(q^3;q)} = q = \frac{\theta(q^3;-q^2)}{\theta(q^3;-q)}. \tag{334}$$

Proof. We show both equalities in (334) together by requiring that each ± below be always simultaneously + or always simultaneously −. One has

$$\theta(q^3;\pm q^2) = \theta\left(q^3;q^3\left(\frac{\pm 1}{q}\right)\right) = q^3\left(\frac{\pm 1}{q}\right)\theta\left(q^3;\frac{\pm 1}{q}\right) \tag{335}$$

$$= q\left[(\pm q)\theta\left(q^3;\frac{\pm 1}{q}\right)\right] = q\left[\theta(q^3;\pm q)\right], \tag{336}$$

where the second equality in (335) is given by the left hand equation in (12) and the last equality in (336) is given by the right hand equation in (12). Dividing (335) and (336) through by $\theta(q^3;\pm q)$ gives (334), and the lemma is shown. □

In the next two propositions, we see for $\delta > 3$ that there are numerous cases for which $\mathscr{W}_{\mu,\lambda}(t)$ is not an extension of a given $f_{\mu,\lambda}(t)$.

Proposition 42. *Let the notation of Theorem 23 hold. Let $\delta > 3$ with $\delta \neq 0 \bmod 4$ and $\delta = 0 \bmod \beta$ be as in (337). Recall that for given $f_{\mu,\lambda}(t)$ one has an integer α and natural numbers β, γ, δ with*

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} \quad \text{in reduced form, and} \quad \frac{\lambda}{2} = \frac{\gamma}{\delta} \quad \text{in reduced form,} \tag{337}$$

while $B = M/\beta = \delta/\beta$, $D = M/\delta = \delta/\delta = 1$, and $\omega = e^{2\pi i/\delta}$. We observe that there are an infinite number of $n \in \mathbb{N}$ such that

$$\frac{\alpha}{\beta} + n\frac{\gamma}{\delta} \notin \mathbb{Z}. \tag{338}$$

Furthermore, recall that

$$\{+\} = \left\{\ell \mid \mathscr{R}\left(\left[\omega^{\gamma}\right]^{\ell+1}\right) > 0\right\} \subset \{0, 1, 2 \cdots (\delta-1)\}. \tag{339}$$

If for any such n as in (338), one has that

$$\sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha+n\gamma} \neq 1, \tag{340}$$

then

$$\mathscr{W}_{\mu,\lambda}(t)\big|_{[0,\infty)} \neq f_{\mu,\lambda}(t). \tag{341}$$

That is, the function $\mathscr{W}_{\mu,\lambda}(t)$ naturally generated by $f_{\mu,\lambda}(t)$ does not restrict to $f_{\mu,\lambda}(t)$ and is not a canonical extension of $f_{\mu,\lambda}(t)$.

Proof. Note first that there are no two consecutive integers n and $n+1$ with

$$\frac{\alpha}{\beta} + n\frac{\gamma}{\delta} \in \mathbb{Z}, \quad \text{and} \quad \frac{\alpha}{\beta} + (n+1)\frac{\gamma}{\delta} \in \mathbb{Z}, \tag{342}$$

for if there were two such consecutive integers, by subtracting one equation from the other in (342), one would conclude that $\gamma/\delta \in \mathbb{Z}$ also. However, this is impossible as γ/δ is in a reduced form with $\delta > 3$, while γ/δ being both an integer and in a reduced form would require $\delta = 1$. We conclude that there is an infinite number of $n \in \mathbb{N}$ with (338) holding. □

Fix any n satisfying (338). For this n, from (138) and from the discussion between (138) through (144), one concludes

$$\begin{aligned}\mathscr{W}^{(n)}_{\mu,\lambda}(0) &= \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha+nD\gamma}(-1)^n f_{\mu+n\lambda,\lambda}(0)\\ &= \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha+nD\gamma}(-1)^n\theta\left(Q;-Q^{(\mu+n\lambda-1)/2}\right)\\ &= \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha+nD\gamma}(-1)^n\theta\left(Q;-Q^{\alpha/\beta+n\gamma/\delta-1}\right).\end{aligned} \tag{343}$$

Furthermore, for our same fixed n, from the fact that $f'_{\mu,\lambda}(t) = -f_{\mu+\lambda,\lambda}(t)$, [2], one has

$$f^{(n)}_{\mu,\lambda}(0) = (-1)^n f_{\mu+n\lambda,\lambda}(0) = (-1)^n\theta\left(Q;-Q^{(\alpha/\beta)+n(\gamma/\delta)-1}\right). \tag{344}$$

Since (338) holds $\theta(Q;-Q^{(\alpha/\beta)+n(\gamma/\delta)-1}) \neq 0$ by (10). If (343) were to equal (344), then dividing through by $\theta(Q;-Q^{(\alpha/\beta)+n(\gamma/\delta)-1})$ and recalling that $D = 1$ in our setting would require that

$$\sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha+nD\gamma} = \sum_{\{+\}}\left[\omega^{\ell+1}\right]^{B\alpha+n\gamma} = 1, \tag{345}$$

which is disallowed by the hypothesis (340). Thus, if n satisfies (338) and (340), one cannot have equality of (343) and (344), whence $\mathscr{W}_{\mu,\lambda}(t)$ does not restrict to $f_{\mu,\lambda}(t)$ on $[0,\infty)$. The proposition is proven.

We next harness Proposition 42 to show that for λ with $\delta = 5, 6, 7$, none of the functions $\mathscr{W}_{\mu,\lambda}(t)$ restricts to $f_{\mu,\lambda}(t)$ on $[0,\infty)$.

Proposition 43. *Under the same setting, notation, and assumptions as Proposition 42, for $f_{\mu,\lambda}(t)$ with α, β, γ as in (337) but with the restriction that $\delta = 5, 6$, or 7, one has $\mathscr{W}_{\mu,\lambda}(t)$ does not restrict to $f_{\mu,\lambda}(t)$ on $[0,\infty)$. Thus, $f_{\mu,\lambda}(t)$ on $[0,\infty)$ does not have a canonical extension for these values of δ.*

Proof. Let $\delta = 5, 6$ or 7. For given γ, define $\widehat{\gamma}$ to be the unique value with $\gamma = \widehat{\gamma} \bmod \delta$ and $0 < \widehat{\gamma} < \delta - 1$. Observe that in (339) for fixed δ one has $\omega^{\gamma} = e^{2\pi i\gamma/\delta} = e^{2\pi i\widehat{\gamma}/\delta} = \omega^{\widehat{\gamma}}$. Thus,

Table 1: Notation and cases as used in Proposition 44.

δ	$\widehat{\gamma}$	$\{\ell+1\}\equiv\{\sigma,\tau,\delta\}$	$\{+\}$	$\{-\}$
5	1	1,4,5	0,3,4	1,2
	2	2,3,5	1,2,4	0,3
	3	2,3,5	1,2,4	0,3
	4	1,4,5	0,3,4	1,2
6	1	1,5,6	0,4,5	1,2,3
	5	1,5,6	0,4,5	1,2,3
7	1	1,6,7	0,5,6	1,2,3,4
	2	3,4,7	2,3,6	0,1,4,5
	3	2,5,7	1,4,6	0,2,3,5
	4	2,5,7	1,4,6	0,2,3,5
	5	3,4,7	2,3,6	0,1,4,5
	6	1,6,7	0,5,6	1,2,3,4

the set $\{+\}$ associated to γ,δ is identical to the set $\{+\}$ associated to $\widehat{\gamma},\delta$. □

In Table 1, for the given value of δ and associated possible $\widehat{\gamma}$ one has, by inspection, the unique values $1\le\ell+1\le\delta$ with $\mathscr{R}(\omega^{\gamma})^{\ell+1}>0$ that determine the set $\{+\}$. We denote the values $\ell+1$ by σ,τ,δ. This last δ is the same value giving $\lambda/2=\gamma/\delta$, where $\delta=5,6,7$.

For instance, to obtain the second row of the above Table 1, when $\delta=5$ and γ has associated value $\widehat{\gamma}=2$, one has

$$\begin{aligned}\mathscr{R}(\omega^{\gamma})^{\ell+1}&=\mathscr{R}\left(e^{2\pi i\gamma/\delta}\right)^{\ell+1}=\mathscr{R}\left(e^{2\pi i\widehat{\gamma}/5}\right)^{\ell+1}\\&=\mathscr{R}\left(e^{2\pi i2/5}\right)^{\ell+1}>0\end{aligned}\tag{346}$$

precisely when the $\ell+1$ in (346) assume the values $\ell+1=2,3$, or 5, that is, when

$$\begin{aligned}\mathscr{R}\left(e^{2\pi i2/5}\right)^{2}&=\cos(2\pi4/5)>0,\ \mathscr{R}\left(e^{2\pi i2/5}\right)^{3}\\&=\cos(2\pi6/5)>0,\ \text{and}\ \mathscr{R}\left(e^{2\pi i2/5}\right)^{5}\\&=1>0.\end{aligned}\tag{347}$$

In this case, one has $\{\ell+1\}=\{\sigma=2,\tau=3,\delta=5\}$ and then $\{+\}=\{\ell\}=\{2-1,3-1,5-1\}=\{1,2,4\}$; the remaining values of ℓ form $\{-\}=\{0,3\}$.

Now note that, in all cases in Table 1,

$$\begin{aligned}&\sigma+\tau=\delta,\quad \text{whence}\ \forall\ j\in\mathbb{Z}\ \text{one has}\ e^{2\pi i\sigma j/\delta}\\&\quad\text{and}\ e^{2\pi i\tau j/\delta}\ \text{are conjugates},\end{aligned}\tag{348}$$

which follows from the fact that $e^{2\pi i\sigma j/\delta}\cdot e^{2\pi i\tau j/\delta}=e^{2\pi i(\sigma+\tau)j/\delta}=e^{2\pi i(\delta)j/\delta}=1$. We conclude that for all $\delta=5,6,7$ and γ with γ/δ in a reduced form, and for all integers j one has

$$\begin{aligned}\sum_{\{+\}}\left(\omega^{j}\right)^{\ell+1}&=\left(e^{2\pi ij/\delta}\right)^{\sigma}+\left(e^{2\pi ij/\delta}\right)^{\tau}+\left(e^{2\pi ij/\delta}\right)^{\delta}\\&=2\cos(2\pi\sigma j/\delta)+1\neq1.\end{aligned}\tag{349}$$

The inequality in (349) follows from the fact that no integer multiple of $2\pi/\delta$ for $\delta=5,6,7$ is an odd multiple of $\pi/2$.

Now, we have seen in Proposition 42 that for each of $\delta=5,6,7$ there are infinitely many n with (338) holding. Fix such an n, and set $j=B\alpha+n\gamma$ in (349) to obtain

$$\sum_{\{+\}}\left(\omega^{[B\alpha+n\gamma]}\right)^{\ell+1}=\sum_{\{+\}}\left(\omega^{\ell+1}\right)^{[B\alpha+n\gamma]}\neq1,\tag{350}$$

from which we see that both (338) and now (340) hold. We conclude from Proposition 42 that $\mathscr{W}_{\mu,\lambda}(t)$ does not restrict to $f_{\mu,\lambda}(t)$ on $[0,\infty)$ for any λ with $\delta=5,6,7$. Therfore, such $f_{\mu,\lambda}(t)$ do not have canonical extensions. This completes the proof of the proposition.

We remark that for $\delta>8$ and not divisible by 4, results similar to Proposition 43, with more tables analogous to Table 1 (but having many more cases), and with inequalities analogous to (349) (but with multiple cosine terms), should give that canonical extensions are at least rare, if not nonexistent.

6. Examples

The goals of this section are twofold:

(1) To classify those μ and λ for low values of $M=\delta$ such that $f_{\mu,\lambda}(t)$ and $\mathscr{W}_{\mu,\lambda}(t)$ meet the assumptions of Theorem 23 (including $\delta=0\ \mathrm{mod}\ \beta$) and satisfy (180)-(182);

(2) To fill out selections from the examples in goal 1 in detail

Throughout the discussion, the notation given in (96) holds, and $M=\mathrm{lcm}\{\beta,\delta\}=\delta$.

6.1. Classifying μ and λ in Theorem 23 with Low $M=\delta$ Values. Recall that the fractions α/β and γ/δ in (96) are in reduced form.

$[M=\delta=1]$: since $M=\delta=\mathrm{lcm}\{\beta,\delta\}=1$, we conclude $\beta=1=\delta$. This is equivalent to

$$\frac{\mu+1}{2}=\frac{\alpha}{\beta}=\frac{\alpha}{1}\qquad\frac{\lambda}{2}=\frac{\gamma}{\delta}=\frac{\gamma}{1}\qquad M=1.\tag{351}$$

Now (351) holds if and only if $\mu=2\alpha-1$ is an odd integer and $\lambda=2\gamma$ is an even positive integer. To match notation in previous work [2], we relabeled $\mu = 2N+1$ and $\lambda = 2n$. Such $f_{2N+1,2n}(t)$ are precisely the $f_{\mu,\lambda}(t)$ that are flat at $t=0$ (as is shown in Proposition 2.2 of [2]). We further analyze this case in detail below and record now that

$$M = \delta = 1 \Leftrightarrow [\mu \in \mathbb{Z} \text{ with } \mu = 1 \bmod 2, \text{ and } \lambda \in \mathbb{N} \text{ with } \lambda = 0 \bmod 2]. \tag{352}$$

[M = δ = p is prime]: since $M = \delta = \operatorname{lcm}\{\beta, \delta\} = p$, we conclude $\beta \in \{1, p\}$ and $\delta = p$. Thus, there are two subcases.

[$M = \delta = p$, $\beta = p$]:

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{p} \Leftrightarrow \mu = \frac{2\alpha - p}{p} \text{ with } \alpha \neq 0 \bmod p,$$
$$\frac{\lambda}{2} = \frac{\gamma}{\delta} = \frac{\gamma}{p} \Leftrightarrow \lambda = \frac{2\gamma}{p} > 0 \text{ with } \gamma \neq 0 \bmod p. \tag{353}$$

[$M = \delta = p$, $\beta = 1$]:

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{1} \Leftrightarrow \mu = 2\alpha - 1 \Leftrightarrow \mu \in \mathbb{Z} \text{ with } \mu = 1 \bmod 2,$$
$$\frac{\lambda}{2} = \frac{\gamma}{\delta} = \frac{\gamma}{p} \Leftrightarrow \lambda = \frac{2\gamma}{p} > 0 \text{ with } \gamma \neq 0 \bmod p. \tag{354}$$

Thus, we have

$$M = \delta = p \text{ is prime} \Leftrightarrow [\text{the rightmost pairs of conditions on } \mu \text{ and } \lambda \text{ in (353) or (354) hold}]. \tag{355}$$

[$M = 4$]: this case is disallowed.

[$M = \delta = 6$]: since $M = \operatorname{lcm}\{\beta, \delta\} = 6$, we conclude $\beta \in \{1, 2, 3, 6\}$ and $\delta = 6$. There are now four subcases.

[$M = \delta = 6$, $\beta = 6$]:

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{6} \Leftrightarrow \mu = \frac{\alpha - 3}{3} \text{ with } \alpha \neq 0 \bmod 2 \text{ and } \alpha \neq 0 \bmod 3,$$
$$\frac{\lambda}{2} = \frac{\gamma}{\delta} = \frac{\gamma}{6} \Leftrightarrow \lambda = \frac{\gamma}{3} > 0 \text{ with } \gamma \neq 0 \bmod 2 \text{ and } \gamma \neq 0 \bmod 3. \tag{356}$$

Please see Section 7 for a detailed convergence study involving the current example (356) with $\alpha = -1$, $\beta = 6$, $\gamma = 1$, and $\delta = 6$.

[$M = \delta = 6$, $\beta = 3$]:

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{3} \Leftrightarrow \mu = \frac{2\alpha - 3}{3} \text{ with } \alpha \neq 0 \bmod 3,$$
$$\frac{\lambda}{2} = \frac{\gamma}{\delta} = \frac{\gamma}{6} \Leftrightarrow \lambda = \frac{\gamma}{3} > 0 \text{ with } \gamma \neq 0 \bmod 2 \text{ and } \gamma \neq 0 \bmod 3. \tag{357}$$

[$M = \delta = 6$, $\beta = 2$]:

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{2} \Leftrightarrow \mu = \alpha - 1 \text{ with } \alpha \neq 0 \bmod 2,$$
$$\frac{\lambda}{2} = \frac{\gamma}{\delta} = \frac{\gamma}{6} \Leftrightarrow \lambda = \frac{\gamma}{3} > 0 \text{ with } \gamma \neq 0 \bmod 2 \text{ and } \gamma \neq 0 \bmod 3. \tag{358}$$

[$M = \delta = 6$, $\beta = 1$]:

$$\frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{1} \Leftrightarrow \mu = 2\alpha - 1 \Leftrightarrow \mu \in \mathbb{Z} \text{ with } \mu = 1 \bmod 2,$$
$$\frac{\lambda}{2} = \frac{\gamma}{\delta} = \frac{\gamma}{6} \Leftrightarrow \lambda = \frac{\gamma}{3} > 0 \text{ with } \gamma \neq 0 \bmod 2 \text{ and } \gamma \neq 0 \bmod 3. \tag{359}$$

Thus, we have

$$M = 6 = \delta \Leftrightarrow \left[\begin{array}{c}\text{any of the rightmost pairs of conditions on} \\ \mu \text{ and } \lambda \text{ in (356) through (359) hold}\end{array}\right]. \tag{360}$$

Other μ and λ with $f_{\mu,\lambda}$ meeting the criteria of Theorem 23 and M not among the above cases can be determined similarly via the following proposition.

Proposition 44. *Let $M = \prod_{j=1}^{J} p_j^{n_j}$ be the prime factorization of $M > 1$ and $M \neq 0 \bmod 4$. The μ and λ with $f_{\mu,\lambda}$ meeting the criteria of Theorem 23 are*

$$\mu = \frac{2\alpha - \beta}{\beta}, \tag{361}$$

$$\lambda = \frac{2\gamma}{\delta} \tag{362}$$

that satisfy the conditions

$$M = \delta = \prod_{j=1}^{J} p_j^{n_j}, \tag{363}$$

$$\beta = \prod_{j=1}^{J} p_j^{k_j} \text{ with } 0 \leq k_j \leq n_j \ \forall j, \tag{364}$$

$$\alpha \neq 0 \bmod p_j \text{ when } k_j > 0, \tag{365}$$

$$\gamma \neq 0 \bmod p_j \ \forall p_j. \tag{366}$$

Proof. From Proposition 21, one sees that the hypothesis that $\delta = 0 \bmod \beta$ in Theorem 23 is required in order to have a non-identically vanishing of (180)–(182). From this, we conclude $M = \operatorname{lcm}\{\beta, \delta\} = \delta$, giving (363). Since β is a divisor of $\delta = M$, one obtains (364). Equations (365) and (366) follow from the fact that α/β and γ/δ are in a reduced form. (361) and (362) follow from (96). The proposition is shown. □

6.2. Selected Examples from Section 6.1 in Further Depth: Canonical Extensions. $[M = \delta = 1,\ \mu = 2N+1,\ \lambda = 2n]$: we expand on [Example $M = 1$] in Section 6.1, where it was shown that $\mu = 2N+1$, $\lambda = 2n$, and $\beta = \delta = 1$, which specifies (96) as

$$\begin{array}{lll} \frac{\mu+1}{2} = \frac{(2N+1)+1}{2} = \frac{N+1}{1} = \frac{\alpha}{\beta} & \frac{\lambda}{2} = \frac{2n}{2} = \frac{n}{1} = \frac{\gamma}{\delta} & M = 1 \\ Q = q^{2/\lambda} = q^{1/n} & B = \frac{M}{\beta} = \frac{1}{1} = 1 & D = \frac{M}{\delta} = 1 \\ & B\alpha = 1\alpha = N+1 & D\gamma = 1n = n \\ \omega = e^{2\pi i/M} = e^{2\pi i} = 1 & & \tilde{\omega} = e^{2\pi i/n} \end{array} \quad (367)$$

Inserting these values in (131) gives (368), while insertion in (180)–(182) gives (369) and (370) below. That is,

$$\mathscr{W}_{2N+1,2n}(t) = \sum_{\ell=0}^{1-1} [1]^N \tilde{f}_{2N+1,2n}(1^n t) = \tilde{f}_{2N+1,2n}(t). \quad (368)$$

Furthermore,

$$\frac{1}{1}\mathscr{F}[\mathscr{W}_{2N+1,2n}(t)](x) = \mathscr{F}\left[\tilde{f}_{2N+1,2n}(t)\right](x) \quad (369)$$

$$= \frac{\mu_Q^3 e^{i\pi(N+1)}}{\sqrt{2\pi}} \frac{1}{(-ix)} \left(\frac{1}{n}\left[\sum_{\kappa=0}^{n-1} \frac{[\tilde{\omega}^\kappa z_3]^{N+1}}{\theta(Q;[\tilde{\omega}^\kappa z_3])}\right]\right) \quad (370)$$

$$= \frac{\mu_Q^3 (-1)^{N+1}}{\sqrt{2\pi}} \frac{1}{(-ix)} \left(\frac{1}{n}\left[\sum_{\kappa=0}^{n-1} \frac{1}{[\tilde{\omega}^\kappa z_3]^{-N-1}\theta(Q;[\tilde{\omega}^\kappa z_3])}\right]\right) \quad (371)$$

$$= \frac{\mu_Q^3 (-1)^{N+1}}{\sqrt{2\pi}} \frac{1}{(-ix)} \cdot \left(\frac{1}{n}\left[\sum_{\kappa=0}^{n-1} \frac{Q^{N(N+1)/2}}{Q^{(-N-1)(-N)/2}[\tilde{\omega}^\kappa z_3]^{-N-1}\theta(Q;[\tilde{\omega}^\kappa z_3])}\right]\right) \quad (372)$$

$$= \frac{\mu_Q^3 (-1)^{N+1}}{\sqrt{2\pi}} \frac{Q^{N(N+1)/2}}{(-ix)} \left(\frac{1}{n}\left[\sum_{\kappa=0}^{n-1} \frac{1}{\theta(Q;[\tilde{\omega}^\kappa z_3]Q^{(-N-1)})}\right]\right), \quad (373)$$

where (371) follows from moving $[\tilde{\omega}^\ell z_3]^{N+1}$ from the numerator to the denominator; (372) is obtained by multiplying numerator and denominator by $Q^{N(N+1)/2}$; and (373) follows from (12). Also, from (184), z_3 in (373) is any fixed n^{th} root of $z_3^n = -e^{-\pi i n} ix$. This example recovers Theorem 6.3 in [2]. Also, if one sets $N = -1$ and $n = 1$, one has $\mathscr{W}_{2N+1,2n}(t) = \mathscr{W}_{-1,2}(t) = \tilde{f}_{-1,2}(t)$ which is discussed earlier in [Example $\mu = -1$ odd and $\lambda = 2$] in Section 5, and which was denoted $K(t)$ in [3]. This recovers the inaugural wavelet in the first paper that inspired the current direction of study.

Finally, $\mathscr{W}_{2N+1,2n}(t) = \tilde{f}_{2N+1,2n}(t)$ satisfies the same multiplied advanced differential equation on $\mathbb{R}$ as does $f_{2N+1,2n}(t)$ on $[0,\infty)$. That is, (185) becomes

$$\mathscr{W}^{(1)}_{2N+1,2n}(t) = (-1)^{n+1} q^{(n+2N+1)/2} \mathscr{W}_{2N+1,2n}(q^n t). \quad (374)$$

$[M = \delta = 2]$: we expand on [Example $M = p$] in the previous section for the case $p = 2$. There are two subcases, when $\beta = 2$ or $\beta = 1$.

$[M = \delta = 2,\ \beta = 2]$:

$$\begin{array}{lll} \frac{\mu+1}{2} = \frac{\alpha}{\beta} = \frac{\alpha}{2} & \frac{\lambda}{2} = \frac{\gamma}{\delta} = \frac{\gamma}{2} & M = 2 \\ Q = q^{2/\lambda} = q^{2/\gamma} & B = \frac{M}{\beta} = \frac{2}{2} = 1 & D = \frac{M}{\delta} = \frac{2}{2} = 1 \\ & B\alpha = 1\alpha = \alpha & D\gamma = 1\gamma = \gamma \\ \omega = e^{2\pi i/2} = e^{\pi i} = -1 & & \tilde{\omega} = e^{2\pi i/[D\gamma]} = e^{2\pi i/\gamma} \end{array} \quad (375)$$

Note that in this case, since α and γ must be odd, $\mu = \alpha - 1$ is even, and $\lambda = \gamma$ is odd, so we set $\mu = 2N$ and $\lambda = 2n + 1 = \gamma$. Then, $B\alpha = 2N+1$ and $D\gamma = 2n+1$. Inserting these values in (131) gives (376)–(378), namely,

$$\mathscr{W}_{2N,2n+1}(t) = [-1]^{2N+1}\tilde{f}_{2N,2n+1}([-1]^{2n+1}t) + \left[[-1]^2\right]^{2N+1}\tilde{f}_{2N,2n+1}\left(\left[(-1)^2\right]^{2n+1}t\right) \quad (376)$$

$$= [-1]\tilde{f}_{2N,2n+1}([-1]t) + [1]\tilde{f}_{2N,2n+1}([1]t) \quad (377)$$

$$= [-1]\left[(-1)f_{2N,2n+1}(-t)\right]\chi_{(-\infty,0)} + \left[f_{2N,2n+1}(t)\right]\chi_{[0,\infty)} \tag{378}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-q^k|t|}}{q^{k(k-2N)/[2n+1]}}. \tag{379}$$

Note that (378) and (379) show that $\mathscr{W}_{2N,2n+1}(t)$ is an even function. Also, from (180)-(182) along with (184), one has

$$\begin{aligned}\mathscr{F}[\mathscr{W}_{2N,2n+1}(t)](x) &= \frac{2\mu_Q^3 e^{i\pi(2N+1)/2}}{\sqrt{2\pi}} \frac{1}{(-ix)} \\ &\cdot \left(\frac{1}{2n+1}\left[\sum_{\kappa=0}^{2n} \frac{\left[e^{2\pi i\kappa/(2n+1)}z_3\right]^{2N+1}}{\theta\left(Q;\left[e^{2\pi i\kappa/(2n+1)}z_3\right]^2\right)}\right]\right),\end{aligned} \tag{380}$$

with

$$z_3^{2n+1} = -e^{-\pi i(2n+1)/2} ix, \tag{381}$$

where $Q = q^{2/(2n+1)}$. Finally, the MADE (185) becomes

$$\begin{aligned}\mathscr{W}^{(2)}_{2N,2n+1}(t) &= (-1)^{2n+1+2} q^{(2n+1)(2n+1+2N)/(2n+1)} \mathscr{W}_{2N,2n+1}\left(q^{2n+1}t\right) \\ &= -q^{2n+1+2N} \mathscr{W}_{2N,2n+1}\left(q^{2n+1}t\right).\end{aligned} \tag{382}$$

When $n = 0$, the above results recover the even case of Theorem 6.5 of [2].

When $n = 0$ and $N = 0$, the above results give

$$\mathscr{W}_{0,1}(t) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-q^k|t|}}{q^{k(k)/[1]}}, \tag{383}$$

with

$$\mathscr{W}_{0,1}(0) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{q^{k^2}} = \sum_{k=-\infty}^{\infty} (-1)^k \frac{q^{-k}}{(q^2)^{k(k-1)/2}} \tag{384}$$

$$= \theta\left(q^2; -\frac{1}{q}\right) = \mu_{q^2} \prod_{n=0}^{\infty} \left(1 - \frac{1}{q^{2n+1}}\right)^2 > 0. \tag{385}$$

In this setting (383)–(385) we recover [Example $\mu = 0$ even and $\lambda = 1$] from Section 5, with normalization $\mathscr{W}_{0,1}(t)/\mathscr{W}_{0,1}(0) = {}_q\mathrm{Cos}(t)$.

$[M = \delta = 2,\ \beta = 1]$:

$$\begin{array}{lll} \dfrac{\mu+1}{2} = \dfrac{\alpha}{\beta} = \dfrac{\alpha}{1} & \dfrac{\lambda}{2} = \dfrac{\gamma}{\delta} = \dfrac{\gamma}{2} & M = 2 \\ Q = q^{2/\lambda} = q^{2/\gamma} & B = \dfrac{M}{\beta} = \dfrac{2}{1} = 2 & D = \dfrac{M}{\delta} = \dfrac{2}{2} = 1 \\ & B\alpha = 2\alpha & D\gamma = 1\gamma = \gamma \\ \omega = e^{2\pi i/2} = e^{\pi i} & & \tilde{\omega} = e^{2\pi i/[D\gamma]} = e^{2\pi i/\gamma} \end{array}. \tag{386}$$

Note that in this case $\mu = 2\alpha - 1$ is odd, and $\lambda = \gamma$ is odd, so we set $\mu = 2N + 1$ and $\lambda = 2n + 1 = \gamma$. Then, $B\alpha = 2(N + 1)$ and $D\gamma = 2n + 1$. Inserting these values in (131) gives (387)–(390), namely,

$$\mathscr{W}_{2N+1,2n+1}(t) = [-1]^{2(N+1)} \tilde{f}_{2N+1,2n+1}\left([-1]^{2n+1}t\right) \tag{387}$$

$$+ \left[[-1]^2\right]^{2(N+1)} \tilde{f}_{2N+1,2n+1}\left(\left[(-1)^2\right]^{2n+1}t\right) \tag{388}$$

$$= [1]\tilde{f}_{2N+1,2n+1}([-1]t) + [1]\tilde{f}_{2N+1,2n+1}([1]t) \tag{389}$$

$$= [1]\left[(-1)f_{2N+1,2n+1}(-t)\right]\chi_{(-\infty,0)} + \left[f_{2N+1,2n+1}(t)\right]\chi_{[0,\infty)} \tag{390}$$

$$= \mathrm{sign}\ (t) \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-q^k|t|}}{q^{k(k-2N-1)/[2n+1]}}. \tag{391}$$

Note that (390) and (391) show that $\mathscr{W}_{2N+1,2n+1}(t)$ is an odd function. Also, from (180)–(182) along with (184), one has

$$\begin{aligned}\mathscr{F}[\mathscr{W}_{2N+1,2n+1}(t)](x) &= \frac{2\mu_Q^3 e^{i\pi(N+1)}}{\sqrt{2\pi}} \frac{1}{(-ix)} \left(\frac{1}{2n+1}\left[\sum_{\kappa=0}^{2n} \frac{\left[e^{2\pi i\kappa/(2n+1)}z_3\right]^{2(N+1)}}{\theta\left(Q;\left[e^{2\pi i\kappa/(2n+1)}z_3\right]^2\right)}\right]\right),\end{aligned} \tag{392}$$

with

$$z_3^{2n+1} = -e^{-\pi i(2n+1)/2} ix, \tag{393}$$

where $Q = q^{2/(2n+1)}$. Finally, the MADE (185) becomes

$$\begin{aligned}\mathscr{W}^{(2)}_{2N+1,2n+1}(t) &= (-1)^{2n+1+2} q^{(2n+1)(2n+1+2N+1)/(2n+1)} \mathscr{W}_{2N+1,2n+1}\left(q^{2n+1}t\right) \\ &= -q^{2n+2+2N} \mathscr{W}_{2N+1,2n+1}\left(q^{2n+1}t\right).\end{aligned} \tag{394}$$

When $n=0$, the above results recover the odd case of Theorem 6.5 of [2].

When $n=0$ and $N=0$, the above results give

$$\mathscr{W}_{1,1}(t)=\operatorname{sign}(t)\sum_{k=-\infty}^{\infty}(-1)^{k}\frac{e^{-q^{k}|t|}}{q^{k(k-1)/[1]}}. \tag{395}$$

In this setting, (395) recovers [Example $\mu=1$ odd and $\lambda=1$] from Section 5, with normalization $\mathscr{W}_{1,1}(t)/\mathscr{W}_{0,1}(0)={}_{q}\mathrm{Sin}(t)$.

[$M=\delta=3$]: we expand on [Example $M=p$] in the previous section for the case $p=3$. There are two subcases.

[$M=\delta=3$, $\beta=3$]:

$$\begin{array}{lll}\dfrac{\mu+1}{2}=\dfrac{\alpha}{\beta}=\dfrac{\alpha}{3} & \dfrac{\lambda}{2}=\dfrac{\gamma}{\delta}=\dfrac{\gamma}{3} & M=3\\ Q=q^{2/\lambda}=q^{3/\gamma} & B=\dfrac{M}{\beta}=\dfrac{3}{3}=1 & D=\dfrac{M}{\delta}=\dfrac{3}{3}=1\\ & B\alpha=1\alpha=\alpha & D\gamma=1\gamma=\gamma\\ \omega=\mathrm{e}^{2\pi i/3} & & \tilde{\omega}=\mathrm{e}^{2\pi i/[D\gamma]}=\mathrm{e}^{2\pi i/[\gamma]}\end{array}. \tag{396}$$

Note that $\mu=2\alpha/3-1$ with $\alpha\neq 0 \bmod 3$ and $\lambda=2\gamma/3$ with $\gamma\neq 0 \bmod 3$. Thus, we set $\alpha=3N+J$ and $\gamma=3n+j$ where $J,j\in\{1,2\}$. Hence, $\mu=2N-1+2J/3$ and $\lambda=2(3n+j)/3$. Then, $B\alpha=3N+J$ and $D\gamma=3n+j$ where $J,j\in\{1,2\}$. Inserting these values in (131) gives (397)–(400), namely,

$$\begin{aligned}\mathscr{W}_{2N-1+[2J/3],2(3n+j)/3}(t)&=\left[\mathrm{e}^{2\pi i/3}\right]^{3N+J}\tilde{f}_{2N-1+[2J/3],2(3n+j)/3}\left(\left[\mathrm{e}^{2\pi i/3}\right]^{3n+j}t\right)+\left[e^{4\pi i/3}\right]^{3N+J}\tilde{f}_{2N-1+[2J/3],2(3n+j)/3}\left(\left[e^{4\pi i/3}\right]^{3n+j}t\right)\\&\quad+[1]^{3N+J}\tilde{f}_{2N-1+[2J/3],2(3n+j)/3}\left([1]^{3n+j}t\right)=\left[e^{2\pi Ji/3}\right]\tilde{f}_{2N-1+[2J/3],2(3n+j)/3}\left(\left[e^{2\pi ji/3}\right]t\right)\end{aligned} \tag{397}$$

$$+\left[e^{4\pi Ji/3}\right]\tilde{f}_{2N-1+[2J/3],2(3n+j)/3}\left(\left[\left(e^{4\pi ji/3}\right]t\right)+[1]\tilde{f}_{2N-1+[2J/3],2(3n+j)/3}\right. \tag{398}$$

$$(t)=\begin{cases}f_{2N-1+[2J/3],2(3n+j)/3}(t), & \text{for } t\geq 0\\ e^{2\pi Ji/3}(-1)f_{2N-1+[2J/3],2(3n+j)/3}\left(\left[e^{2\pi ji/3}\right]t\right)+e^{4\pi Ji/3}(-1)f_{2N-1+[2J/3],2(3n+j)/3}\left(\left[e^{4\pi ji/3}\right]t\right)\Big\}, & \text{for } t<0\end{cases} \tag{399}$$

$$=\begin{cases}\displaystyle\sum_{k=-\infty}^{\infty}\frac{(-1)^{k}\exp\left(-q^{k}t\right)}{q^{k(k-2N+1-[2J/3])/[2(3n+j)/3]}}, & \text{for } t\geq 0,\\ \displaystyle e^{2\pi Ji/3}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}\exp\left(-q^{k}\left[\mathrm{e}^{2\pi ji/3}\right]t\right)}{q^{k(k-2N+1-[2J/3])/[2(3n+j)/3]}}+e^{4\pi Ji/3}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+1}\exp\left(-q^{k}\left[\mathrm{e}^{4\pi ji/3}\right]t\right)}{q^{k(k-2N+1-[2J/3])/[2(3n+j)/3]}}, & \text{for } t<0.\end{cases} \tag{400}$$

Also, from (180)–(182) along with (184), one has:

$$\begin{aligned}&\mathscr{F}\left[\mathscr{W}_{2N-1+[2J/3],2(3n+j)/3}(t)\right](x)\\&\quad=\frac{3\mu_{Q}^{3}e^{i\pi(N+[J/3])}}{\sqrt{2\pi}}\frac{1}{(-ix)}\\&\quad\cdot\left(\frac{1}{3n+j}\left[\sum_{\kappa=0}^{3n+j-1}\frac{\left[e^{2\pi i\kappa/(3n+j)}z_{3}\right]^{3N+J}}{\theta\left(Q;\left[e^{2\pi i\kappa/(3n+j)}z_{3}\right]^{3}\right)}\right]\right),\end{aligned} \tag{401}$$

with

$$z_{3}^{3n+j}=-e^{-\pi i(3n+j)/3}ix, \tag{402}$$

where $Q=q^{3/(3n+j)}$. Finally, the MADE (185) becomes

$$\begin{aligned}&\mathscr{W}^{(3)}_{2N-1+[2J/3],2(3n+j)/3}(t)\\&\quad=(-1)^{3n+j+3}q^{(3n+j+2N-1+[2J/3])/[2/3]}\\&\quad\cdot\mathscr{W}_{2N-1+[2J/3],[2(3n+j)/3]}\left(q^{3n+j}t\right).\end{aligned} \tag{403}$$

[$M=\delta=3$, $\beta=1$]:

$$\begin{array}{lll}\dfrac{\mu+1}{2}=\dfrac{\alpha}{\beta}=\dfrac{\alpha}{1} & \dfrac{\lambda}{2}=\dfrac{\gamma}{\delta}=\dfrac{\gamma}{3} & M=3\\ Q=q^{2/\lambda}=q^{3/\gamma} & B=\dfrac{M}{\beta}=\dfrac{3}{1}=3 & D=\dfrac{M}{\delta}=\dfrac{3}{3}=1\\ & B\alpha=3\alpha & D\gamma=1\gamma=\gamma\\ \omega=e^{2\pi i/3} & & \tilde{\omega}=e^{2\pi i/[D\gamma]}=e^{2\pi i/[\gamma]}\end{array}. \tag{404}$$

Note that $\mu = 2\alpha - 1$ is odd and $\lambda = 2\gamma/3$ with $\gamma \neq 0 \bmod 3$. Thus, we set $\mu = 2N+1$ with $\gamma = 3n+j$ where $j = 1, 2$ and $\lambda = 2(3n+j)/3$. Then, $B\alpha = 3(N+1)$ and $D\gamma = 3n+j$ where $j = 1, 2$. Inserting these values in (131) gives (405)–(408), namely,

$$\begin{aligned}\mathscr{W}_{2N+1,2(3n+j)/3}(t) &= \left[e^{2\pi i/3}\right]^{3(N+1)}\tilde{f}_{2N+1,2(3n+j)/3}\left(\left[e^{2\pi i/3}\right]^{3n+j}t\right) + \left[e^{4\pi i/3}\right]^{3(N+1)}\tilde{f}_{2N+1,2(3n+j)/3}\left(\left[e^{4\pi i/3}\right]^{3n+j}t\right)\\ &\quad + [1]^{3(N+1)}\tilde{f}_{2N+1,2(3n+j)/3}\left([1]^{3n+j}t\right)\end{aligned} \tag{405}$$

$$= [1]\tilde{f}_{2N+1,2(3n+j)/3}\left(\left[e^{2\pi ji/3}\right]t\right) + [1]\tilde{f}_{2N+1,2(3n+j)/3}\left(\left[\left(e^{4\pi ji/3}\right]t\right) + [1]\tilde{f}_{2N+1,2(3n+j)/3}(t) \tag{406}$$

$$= \begin{cases} f_{2N+1,2(3n+j)/3}(t), & \text{for } t \geq 0\\ (-1)f_{2N+1,2(3n+j)/3}\left(\left[e^{2\pi ji/3}\right]t\right) + (-1)f_{2N+1,2(3n+j)/3}\left(\left[e^{4\pi ji/3}\right]t\right)\Big\}, & \text{for } t < 0\end{cases} \tag{407}$$

$$= \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty}(-1)^k\frac{\exp\left(-q^k t\right)}{q^{k(k-2N-1)/[2(3n+j)/3]}} & \text{for } t \geq 0,\\ \displaystyle(-1)\sum_{k=-\infty}^{\infty}(-1)^k\frac{\exp\left(-q^k\left[e^{2\pi i/3}\right]t\right)}{q^{k(k-2N-1)/[2(3n+j)/3]}} + (-1)\sum_{k=-\infty}^{\infty}(-1)^k\frac{\exp\left(-q^k\left[e^{4\pi i/3}\right]t\right)}{q^{k(k-2N-1)/[2(3n+j)/3]}}, & \text{for } t < 0,\end{cases} \tag{408}$$

where each of the three summations in (408) vanishes at $t = 0$ as one can compute that

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{q^{k(k-2N-1)/[2(3n+j)/3]}} = \theta\left(Q;-Q^N\right) = 0, \tag{409}$$

with Q as in (404), where the vanishing follows from (13).

Also, from (180)–(182) along with (184) one has

$$\begin{aligned}\mathscr{F}&\left[\mathscr{W}_{2N+1,2(3n+j)/3}(t)\right](x)\\ &= \frac{3\mu_Q^3 e^{i\pi(N+1)}}{\sqrt{2\pi}}\frac{1}{(-ix)}\\ &\cdot\left(\frac{1}{3n+j}\left[\sum_{\kappa=0}^{3n+j-1}\frac{\left[e^{2\pi i\kappa/(3n+j)}z_3\right]^{3(N+1)}}{\theta\left(Q;\left[e^{2\pi i\kappa/(3n+j)}z_3\right]^3\right)}\right]\right),\end{aligned} \tag{410}$$

with

$$z_3^{3n+j} = -e^{-\pi i(3n+j)/3}ix, \tag{411}$$

where $Q = q^{3/(3n+j)}$. Finally, the MADE (185) becomes

$$\begin{aligned}\mathscr{W}&^{(3)}_{2N+1,[2(3n+j)/3]}(t)\\ &= (-1)^{3n+j+3}q^{(3n+j+2N+1)/[2/3]}\mathscr{W}_{2N+1,[2(3n+j)/3]}\left(q^{3n+j}t\right).\end{aligned} \tag{412}$$

When $N = 0$, $n = 0$, and $j = 1$, equations (410) and (412) recover, respectively, the Fourier transform and the MADE satisfied by $\mathscr{W}_{1,2/3}(t)$ as seen in [Example $\mu = 1$ odd and $\lambda = 2/3$] in Section 5.

7. Convergence of MADEs to Classical Solutions, an Example

The purpose of this section is to provide an overview of an example of a $\mathscr{W}_{\mu,\lambda}(t)$ satisfying a MADE where the normalization $\mathscr{W}_{\mu,\lambda}(t)/\mathscr{W}_{\mu,\lambda}(0)$ converges to the corresponding solution of its classical analogue (the ODE obtained when the parameter q in the original MADE is set to 1). Our example is $\mathscr{W}_{-4/3,1/3}(t)$, which is not a canonical extensions as will be seen below. As such, it represents a new phenomenon with corresponding new challenges. First, $\mathscr{W}_{-4/3,1/3}(t)$ can be obtained as the output of a reproducing kernel computation for an input wavelet $\mathscr{W}_{-1,2/3}(t)$, in the same manner of computation as in [4]. Second, since it satisfies a MADE of order 6, there are 6 derivatives to determine and use to compute initial conditions at the origin. Third, determining the initial conditions for the analogous ODE will be accomplished with the aide of a significant new result, namely the generalized q-Wallis formulas in Theorem 46. This theorem is also crucial in our proof of convergence of the normalized solution of the MADE to the solution of the analogous ODE. The normalization $\mathscr{W}_{-4/3,1/3}(t)/\mathscr{W}_{-4/3,1/3}(0)$ will be our main object of study as the parameter $q \longrightarrow 1^+$. From (425), the parameters $\mu = -4/3$ and $\lambda = 1/3$ show our example $\mathscr{W}_{-4/3,1/3}(t)$ to be from [$M = \delta = 6$, $\beta = 6$] in Section 6.

7.1. Preliminaries. For $b \geq 0$, the reproducing kernel computation relevant to our setting is

$$\int_{-\infty}^{\infty} \mathscr{W}_{-1,2/3}(t)\mathscr{W}_{-1,2/3}(t-b)\,dt \tag{413}$$

$$= \frac{-3i\left(\mu_{q^3}\right)^4}{2\left(\mu_{q^6}\right)^2} \cdot \sum_{\ell=0}^{2}\left\{\left[\exp\left(\frac{2\pi i\ell+\pi i}{6}\right)\right]^{-1} f_{-4/3,1/3} \cdot \left(-i\exp\left(\frac{2\pi i\ell+i\pi}{6}\right)b\right)\right\} \tag{414}$$

$$= \frac{-3i\left(\mu_{q^3}\right)^4}{2\left(\mu_{q^6}\right)^2}(-i)\cdot\left[e^{2\pi i/6} f_{-4/3,1/3}\left(e^{-2\pi i/6}b\right) + f_{-4/3,1/3}(b) + e^{-2\pi i/6} f_{-4/3,1/3}\left(e^{2\pi i/6}b\right)\right] \tag{415}$$

$$= \frac{-3i\left(\mu_{q^3}\right)^4}{2\left(\mu_{q^6}\right)^2}(-i)\mathscr{W}_{-4/3,1/3}(b). \tag{416}$$

Moving from (413) to (414) is accomplished via Plancherel's theorem where the expressions for the Fourier transform of $\mathscr{W}_{-1,2/3}$ are given by Theorem 23. The resulting integral of Fourier transforms is evaluated with a residue computation in the upper half-plane (for $b \geq 0$) similar in nature to the computation of reproducing kernels for $b \geq 0$ in [4]. This computation is lengthy, and the details are left as part of a more general set of reproducing kernel computations in an upcoming work. A direct computation moves one from line (414) to (415), while equation (132) of Definition 16 gives for $b \geq 0$ that

$$\mathscr{W}_{-4/3,1/3}(b) = \left[e^{2\pi i/6} f_{-4/3,1/3}\left(e^{-2\pi i/6}b\right) + f_{-4/3,1/3}(b) + e^{-2\pi i/6} f_{-4/3,1/3}\left(e^{2\pi i/6}b\right)\right], \tag{417}$$

which justifies movement from (415) to (416).

Setting $b = 0$ in (413)–(416) yields

$$\int_{-\infty}^{\infty} \mathscr{W}_{-1,2/3}(t)\mathscr{W}_{-1,2/3}(t)\,dt = \left\|\mathscr{W}_{-1,2/3}(t)\right\|_2^2 = \frac{-3i\left(\mu_{q^3}\right)^4}{2\left(\mu_{q^6}\right)^2}(-i)\mathscr{W}_{-4/3,1/3}(0), \tag{418}$$

which in turn gives the functional identity

$$\frac{1}{\mathscr{W}_{-4/3,1/3}(0)} = \frac{-3\left(\mu_{q^3}\right)^4}{2\left(\mu_{q^6}\right)^2\left\|\mathscr{W}_{-1,2/3}(t)\right\|_2^2}. \tag{419}$$

Normalizing (413)–(416) by the squared $\mathscr{L}^2$ norm and applying the functional identity in (419), one observes that

$$\frac{1}{\left\|\mathscr{W}_{-1,2/3}(t)\right\|_2^2}\int_{-\infty}^{\infty} \mathscr{W}_{-1,2/3}(t)\mathscr{W}_{-1,2/3}(t-b)\,dt = \frac{-3\left(\mu_{q^3}\right)^4}{2\left(\mu_{q^6}\right)^2\left\|\mathscr{W}_{-1,2/3}(t)\right\|_2^2}\mathscr{W}_{-4/3,1/3}(b) = \frac{\mathscr{W}_{-4/3,1/3}(b)}{\mathscr{W}_{-4/3,1/3}(0)}. \tag{420}$$

Recalling that $\mathscr{W}_{-4/3,1/3}(t)$ is real valued, and applying Cauchy-Schwartz to (420) yields

$$\begin{aligned} 1 &= \left\|\frac{\mathscr{W}_{-1,2/3}(t)}{\left\|\mathscr{W}_{-1,2/3}(t)\right\|_2}\right\|_2\left\|\frac{\mathscr{W}_{-1,2/3}(t-b)}{\left\|\mathscr{W}_{-1,2/3}(t)\right\|_2}\right\|_2 \\ &\geq \left|\frac{1}{\left\|\mathscr{W}_{-1,2/3}(t)\right\|_2^2}\int_{-\infty}^{\infty}\mathscr{W}_{-1,2/3}(t)\mathscr{W}_{-1,2/3}(t-b)dt\right| \\ &= \left|\frac{\mathscr{W}_{-4/3,1/3}(b)}{\mathscr{W}_{-4/3,1/3}(0)}\right|, \end{aligned} \tag{421}$$

which gives a unit global bound on the normalization $|\mathscr{W}_{-4/3,1/3}(b)/\mathscr{W}_{-4/3,1/3}(0)|$ independent of q for all $b \geq 0$. Now, equation (132) of Definition 16 also gives for $b < 0$ that

$$\begin{aligned} \mathscr{W}_{-4/3,1/3}(b) &= \left[e^{-4\pi i/6}(-1) f_{-4/3,1/3}\left(e^{4\pi i/6}b\right) + e^{-6\pi i/6}(-1) \cdot f_{-4/3,1/3}\left(e^{6\pi i/6}b\right) + e^{-8\pi i/6}(-1) f_{-4/3,1/3}\left(e^{8\pi i/6}b\right)\right] \\ &= \left[e^{2\pi i/6} f_{-4/3,1/3}\left(e^{-2\pi i/6}(-1)b\right) + f_{-4/3,1/3}((-1)b) + e^{-2\pi i/6} f_{-4/3,1/3}\left(e^{2\pi i/6}(-1)b\right)\right] \\ &= \left[e^{2\pi i/6} f_{-4/3,1/3}\left(e^{-2\pi i/6}|b|\right) + f_{-4/3,1/3}(|b|) + e^{-2\pi i/6} f_{-4/3,1/3}\left(e^{2\pi i/6}|b|\right)\right] \end{aligned} \tag{422}$$

$$= \mathscr{W}_{-4/3,1/3}(|b|), \tag{423}$$

where comparison of (422) with (417) allows movement to (423). So $\mathscr{W}_{-4/3,1/3}(b)$ is an even Schwartz function and consequently the bound in (421) extends to a global bound: for all $b \in \mathbb{R}$ and for each $q > 1$, one has

$$1 \geq \left|\frac{\mathscr{W}_{-4/3,1/3}(b)}{\mathscr{W}_{-4/3,1/3}(0)}\right|. \tag{424}$$

From (415) and (416), one sees that $\mathscr{W}_{-4/3,1/3}(t)$ satisfies the same MADE as does $f_{-4/3,1/3}(t)$, where $\mu = -4/3$ and $\lambda = 1/3$. From these values, we deduce that

$$\begin{aligned}\frac{\mu+1}{2} &= \frac{-4/3+1}{2} = \frac{-1/3}{2} = \frac{-1}{6} \\ &= \frac{\alpha}{\beta}, \quad \frac{\lambda}{2} = \frac{1/3}{2} = \frac{1}{6} = \frac{\gamma}{\delta},\end{aligned} \tag{425}$$

by which one sees that $\alpha = -1, \beta = 6, \gamma = 1, \delta = 6$. From these parameter values, one sees that $\mathscr{W}_{-4/3,1/3}(t)$ falls in the example class $[M = \delta = 6, \beta = 6]$ with (356) holding. Since $\delta = 6$, we conclude from Proposition 43 that $\mathscr{W}_{-4/3,1/3}(t)$ is not a canonical extension of $f_{-4/3,1/3}(t)$. See also Table 1 where $\mathscr{W}_{-4/3,1/3}(t)$ fall in the category of row $\delta = 6, \widehat{\gamma} = 1, \{\ell+1\} = \{1,5,6\}, \{+\} = \{0,4,5\}$, and $\{-\} = \{1,2,3\}$. From (2), one determines the MADE for $f_{-4/3,1/3}(t)$ to be

$$\begin{aligned}f^{(6)}_{-4/3,1/3}(t) &= (-1)^{1+6} q^{1(1+(-4/3))/(1/3)} f_{-4/3,1/3}(q^1 t) \\ &= -q^{-1} f_{-4/3,1/3}(qt).\end{aligned} \tag{426}$$

And hence,

$$\mathscr{W}^{(6)}_{-4/3,1/3}(t) = -q^{-1}\mathscr{W}_{-4/3,1/3}(qt), \tag{427}$$

while

$$\left[\frac{\mathscr{W}_{-4/3,1/3}(t)}{\mathscr{W}_{-4/3,1/3}(0)}\right]^{(6)} = -q^{-1}\left[\frac{\mathscr{W}_{-4/3,1/3}(qt)}{\mathscr{W}_{-4/3,1/3}(0)}\right]. \tag{428}$$

Each of (426)–(428) we take to be our MADE under study for this example. We take the classical analogue of our MADE to be the ODE obtained by setting $q = 1$ in the original MADE. In our case, the classical analogue of (426)–(428) in this example is

$$g^{(6)}(t) = -g(t). \tag{429}$$

Next, we turn to the computation of our derivatives, which will in turn lead to the initial conditions for our MADE and (by taking the limit as $q \longrightarrow 1^+$) the initial conditions of our analogous ODE (429). Relying on (417) and the fact that

$$f_{\mu,\lambda}'(t) = -f_{\mu+\lambda,\lambda}(t), \tag{430}$$

(which can be directly computed from (1)), one sees that

$$\begin{aligned}\mathscr{W}^{(m)}_{-4/3,1/3}(t) &= \Big[e^{2\pi i/6} f^{(m)}_{-4/3,1/3}\left(e^{-2\pi i/6} t\right)\left[e^{-2\pi i/6}\right]^m + f^{(m)}_{-4/3,1/3}(t) \\ &\quad + e^{-2\pi i/6} f^{(m)}_{-4/3,1/3}\left(e^{2\pi i/6} t\right)\left[e^{2\pi i/6}\right]^m\Big] \\ &= (-1)^m \Big[f_{-4/3+m/3,1/3}\left(e^{-2\pi i/6} t\right) e^{-(m-1)2\pi i/6} \\ &\quad + f_{-4/3+m/3,1/3}(t) + f_{-4/3+m/3,1/3}\left(e^{2\pi i/6} t\right) e^{(m-1)2\pi i/6}\Big].\end{aligned} \tag{431}$$

When $m = 6$, we have a direct check that the MADE given in (427) holds, after noting that $f_{-4/3+2,1/3}(t) = -q^{-1} f_{-4/3,1/3}(qt)$ which follows from (441) below with $\ell = 1$. Evaluating (431) at $t = 0$ yields

$$\begin{aligned}&\mathscr{W}^{(m)}_{-4/3,1/3}(0) \\ &\quad = (-1)^m \left[e^{-(m-1)2\pi i/6} + 1 + e^{(m-1)2\pi i/6}\right] f_{-4/3+m/3,1/3}(0) \\ &\quad = (-1)^m [1 + 2\cos((m-1)2\pi/6)] f_{-4/3+m/3,1/3}(0) \\ &\quad = (-1)^m [1 + 2\cos((m-1)2\pi/6)]\,\theta\left(q^6; -1\left(q^3\right)^{-7/3+m/3}\right),\end{aligned} \tag{432}$$

where we have used the fact that

$$\begin{aligned}f_{\mu,\lambda}(0) &= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k\mu/\lambda} q^{-k/\lambda}}{q^{k(k)/\lambda} q^{-k/\lambda}} \\ &= \sum_{k=-\infty}^{\infty} \frac{\left[-q^{(\mu-1)/\lambda}\right]^k}{\left(q^{2/\lambda}\right)^{k(k-1)/2}} = \theta\left(q^{2/\lambda}; -q^{(\mu-1)/\lambda}\right),\end{aligned} \tag{433}$$

to move to (432). Note that the last equality in (433) follows from (10).

Setting $m = 6\ell + j$ with $j \in \{0, 1, 2, \cdots, 5\}$ in (432) and relying on (12), it follows that

$$\begin{aligned}\mathscr{W}^{(6\ell+j)}_{-4/3,1/3}(0) &= (-1)^{6\ell+j}[1 + 2\cos((6\ell + j - 1)2\pi/6)]\,\theta \\ &\quad \cdot \left(q^6; -1\left(q^3\right)^{-7/3+(6\ell+j)/3}\right) \\ &= (-1)^j [1 + 2\cos((j-1)2\pi/6)]\,\theta \\ &\quad \cdot \left(q^6; -1 q^{6(\ell-1)+j-1}\right) \\ &= (-1)^j [1 + 2\cos((j-1)2\pi/6)]\, q^{6(\ell-1)\ell/2} \\ &\quad \cdot \left(-q^{j-1}\right)^{\ell-1} \theta\left(q^6; -q^{j-1}\right).\end{aligned} \tag{434}$$

From (434), we see that for $j = 1$ the theta factor $\theta(q^6; -q^{j-1}) = \theta(q^6; -1) = 0$ by (13). Also from (414), for $j = 3, 5$ one has $[1 + 2\cos((j-1)2\pi/6)] = 0$. We conclude that $\mathscr{W}^{(6\ell+j)}_{-4/3,1/3}(0) = 0$ for $j = 1, 3, 5$ odd. Setting $\ell = 0, j = 0$ in (434) gives that $\mathscr{W}_{-4/3,1/3}(0) = [1 + 2\cos((-1)2\pi/6)](-q^{-1})^{-1}\theta(q^6; -q^{-1})$

$= -2q\theta(q^6;-q^{-1})$. Thus, the derivatives of all orders of the normalized function at $t=0$ are expressed via

$$\frac{\mathscr{W}^{(6\ell+j)}_{-4/3,1/3}(0)}{\mathscr{W}_{-4/3,1/3}(0)} = \begin{cases} 0, & j=1,3,5 \\ \dfrac{(-1)^{\ell}[1+2\cos((j-1)2\pi/6)]}{2} q^{3(\ell-1)\ell} q^{(j-1)(\ell-1)-1} \dfrac{\theta(q^6;-q^{j-1})}{\theta(q^6;-q^{-1})}, & j=0,2,4 \end{cases} \tag{435}$$

$$= \begin{cases} 0, & j=1,3,5 \\ \dfrac{(-1)^{\ell} q^{3(\ell-1)\ell} q^{(-1)(\ell-1)-1}\theta(q^6;-q^{-1})}{\theta(q^6;-q^{-1})}, & j=0 \\ \dfrac{(-1)^{\ell} q^{3(\ell-1)\ell} q^{(1)(\ell-1)-1}\theta(q^6;-q^{1})}{\theta(q^6;-q^{-1})}, & j=2 \\ \dfrac{(-1)^{\ell}/2[-1] q^{3(\ell-1)\ell} q^{3(\ell-1)-1}\theta(q^6;-q^{3})}{\theta(q^6;-q^{-1})}, & j=4 \end{cases} \tag{436}$$

$$= \begin{cases} 0, & j=1,3,5 \\ (-1)^{\ell} q^{3(\ell-1)\ell} q^{(-1)(\ell-1)-1}, & j=0 \\ (-1)^{\ell} q^{3(\ell-1)\ell} q^{(1)(\ell-1)-1}(-q), & j=2 \\ \dfrac{(-1)^{\ell}/2[-1] q^{3(\ell-1)\ell} q^{3(\ell-1)-1}\theta(q^6;-q^{3})}{\theta(q^6;-q^{-1})}, & j=4 \end{cases} \tag{437}$$

where moving to the $j=2$ case in (437) is facilitated by the fact that $-q\theta(q^6;-q^{-1}) = \theta(q^6;-q^1)$ which is obtained from the right hand equation in (12) when q is replaced by q^6 and u is replaced by $-q$. We remark that, from (430) and (433), the derivatives of all orders of all normalized $f_{\mu,\lambda}(t)$ (and functions constructed from the $f_{\mu,\lambda}(t)$ such as $\mathscr{W}_{\mu,\lambda}(t)$) at $t=0$ are expressible in terms of ratios of theta functions, as is seen overtly in (436) above for the normalized $\mathscr{W}_{\mu,\lambda}(t)/\mathscr{W}_{\mu,\lambda}(0)$.

We next determine the derivatives of all orders for the ODE in (429) analogous to our MADE in (428) by taking the limits as $q \longrightarrow 1^+$ of the derivatives obtained in (437). That is

$$g^{(6\ell+j)}(0) \equiv \lim_{q\longrightarrow 1^+} \frac{\mathscr{W}^{(6\ell+j)}_{-4/3,1/3}(0)}{\mathscr{W}_{-4/3,1/3}(0)} = \begin{cases} 0, & j=1,3,5 \\ \lim\limits_{q\longrightarrow 1^+} (-1)^{\ell} q^{3(\ell-1)\ell} q^{(-1)(\ell-1)-1}, & j=0 \\ \lim\limits_{q\longrightarrow 1^+} (-1)^{\ell} q^{3(\ell-1)\ell} q^{(1)(\ell-1)-1}(-q), & j=2 \\ \lim\limits_{q\longrightarrow 1^+} \dfrac{(-1)^{\ell}}{2}[-1] q^{3(\ell-1)\ell} q^{3(\ell-1)-1} \dfrac{\theta(q^6;-q^3)}{\theta(q^6;-q^{-1})}, & j=4 \end{cases} \tag{438}$$

$$= \begin{cases} 0, & j=1,3,5 \\ (-1)^{\ell}, & j=0 \\ (-1)^{\ell}(-1), & j=2 \\ \dfrac{(-1)^{\ell}}{2}[-1] \lim\limits_{q\longrightarrow 1^+} \dfrac{\theta(q^6;-q^3)}{\theta(q^6;-q^{-1})} = \dfrac{(-1)^{\ell}}{2}[-1][-2] = (-1)^{\ell}, & j=4 \end{cases} \tag{439}$$

where we have relied on the power of the generalized q-Wallis formulas for ratios of theta functions obtained in Section 7.3 and given by (470), (482), and (485) below to evaluate the limit in the case $j=4$ of (438) above. More precisely, by (482) in the case at hand, we have

$$\lim_{q\longrightarrow 1^+} \left\{ \frac{\theta(q^6;-q^3)}{\theta(q^6;-q^{-1})} \right\} = \lim_{q\longrightarrow 1^+} \left\{ \frac{\theta(q^6;-q^{6(0)+3})}{\theta(q^6;-q^{6(-1)+5})} \right\} = (-1)^{0-1} \frac{\sin(3\pi/6)}{\sin(5\pi/6)} = -\frac{1}{(1/2)} = [-2]. \tag{440}$$

One final observation regarding higher order derivatives of order 0 mod 6 follows. From (430), one has

$$\begin{aligned} f^{(6\ell)}_{-4/3,1/3}(t) &= (-1)^{6\ell} f_{-4/3+6\ell/3,1/3}(t) = f_{-4/3+2\ell,1/3}(t) \\ &= \sum_{k=-\infty}^{\infty} \frac{(-1)^k e^{-q^k t}}{q^{k(k+4/3-2\ell)/1/3}} = \sum_{k=-\infty}^{\infty} \frac{(-1)^{k-\ell+\ell} e^{-q^{k-\ell}(q^\ell t)}}{q^{(k-\ell+\ell)(k-\ell-\ell+4/3)/1/3}} \\ &= \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+\ell} e^{-q^m(q^\ell t)}}{q^{(m+\ell)(m-\ell+4/3)/1/3}} \\ &= (-1)^\ell \sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{-q^m(q^\ell t)}}{q^{(m^2-\ell^2+(4/3)m+(4/3)\ell)/1/3}} \\ &= (-1)^\ell q^{(\ell^2-4\ell/3)/(1/3)} \sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{-q^m(q^\ell t)}}{q^{(m(m+4/3))/1/3}} \\ &= (-1)^\ell q^{\ell(\ell-4/3)/(1/3)} f_{-4/3,1/3}(q^\ell t). \end{aligned} \tag{441}$$

7.2. Convergence. We now are prepared to prove that $f(t) = \mathscr{W}_{-4/3,1/3}(t)/\mathscr{W}_{-4/3,1/3}(0)$ satisfying the MADE $f^{(6)}(t) = -q^{-1}f(qt)$ converges as $q \longrightarrow 1^+$ to the solution $g(t)$ of the analogous ODE $g^{(6)}(t) = -g(t)$ having initial conditions given by (439) with $\ell = 0$. That is, the initial conditions are given by

$$\begin{aligned} &g(0) = 1, \quad g^{(1)}(0) = 0, \quad g^{(2)}(0) = -1, \\ &g^{(3)}(0) = 0, \quad g^{(4)}(0) = 1, \quad g^{(5)}(0) = 0. \end{aligned} \tag{442}$$

The unique ODE solution satisfying (442) is given by $g(t) = \cos(t)$, by inspection. Convergence of $f(t)$ to $g(t) = \cos(t)$ will be in the sense of uniform convergence on all compact subsets S of $\mathbb{R}$. The proof will hinge on three factors (for K sufficiently large): (1) proximity of $f(t)$ to the Taylor polynomial $P_K[f](t)$ on S; (2) proximity of $g(t)$ to the Taylor polynomial $P_K[g](t)$ on S; and (3) proximity of $P_K[f](t)$ to $P_K[g](t)$ on S. These in turn then force proximity of $f(t)$ to $g(t)$.

Proposition 45. *For any compact set S in $\mathbb{R}$, $f(t) = \mathscr{W}_{-4/3,1/3}(t)/\mathscr{W}_{-4/3,1/3}(0)$ converges uniformly to $g(t) = \cos(t)$ on S as $q \longrightarrow 1^+$.*

Proof. First, the $6N+5$-degree Taylor polynomials $P_{6N+5}[g](t), P_{6N+5}[f](t)$ of g and f, respectively, expanded about $t = 0$ are given by

$$\begin{aligned} P_{6N+5}[g](t) &= \sum_{n=0}^{6N+5} \frac{g^{(n)}(0)}{n!} t^n = \sum_{k=0}^{N} \frac{g^{(6k)}(0)}{(6k)!} t^{6k} \\ &\quad + \sum_{k=0}^{N} \frac{g^{(6k+2)}(0)}{(6k+2)!} t^{6k+2} + \sum_{k=0}^{N} \frac{g^{(6k+4)}(0)}{(6k+4)!} t^{6k+4} \\ &= \sum_{k=0}^{N} \frac{(-1)^k}{(6k)!} t^{6k} + \sum_{k=0}^{N} \frac{(-1)^{k+1}}{(6k+2)!} t^{6k+2} \\ &\quad + \sum_{k=0}^{N} \frac{(-1)^k}{(6k+4)!} t^{6k+4} \end{aligned} \tag{443}$$

$$\begin{aligned} P_{6N+5}[f](t) &= \sum_{n=0}^{6N+5} \frac{f^{(n)}(0)}{n!} t^n = \sum_{k=0}^{N} \frac{f^{(6k)}(0)}{(6k)!} t^{6k} \\ &\quad + \sum_{k=0}^{N} \frac{f^{(6k+2)}(0)}{(6k+2)!} t^{6k+2} + \sum_{k=0}^{N} \frac{f^{(6k+4)}(0)}{(6k+4)!} t^{6k+4} \\ &= \sum_{k=0}^{N} \frac{(-1)^k q^{3k(k-1)} q^{(-1)(k-1)-1}}{(6k)!} t^{6k} \end{aligned} \tag{444}$$

$$+ \sum_{k=0}^{N} \frac{(-1)^k q^{3k(k-1)} q^{(k-1)-1}(-q)}{(6k+2)!} t^{6k+2} \tag{445}$$

$$+ \sum_{k=0}^{N} \frac{(-1)^k q^{3k(k-1)} q^{3(k-1)-1} \left(\theta(q^6;-q^3)/\left[-2\theta(q^6;-q^{-1})\right]\right)}{(6k+4)!} t^{6k+4}, \tag{446}$$

where (443) follows from (439), and (444)–(446) follow from (437). These have respective remainder terms

$$R_{6N+5}[g](t) = \frac{g^{(6N+6)}(\xi)}{(6N+6)!} t^{6N+6} = \frac{(-1)^{N+1} g(\xi)}{(6N+6)!} t^{6N+6}, \tag{447}$$

for some ξ between 0 and t, and

$$\begin{aligned} R_{6N+5}[f](t) &= \frac{f^{(6N+6)}(\zeta)}{(6N+6)!} t^{6N+6} \\ &= \frac{(-1)^{N+1} q^{(N+1)(N+1-4/3)/(1/3)} f(q^{N+1}\zeta)}{(6N+6)!} t^{6N+6}, \end{aligned} \tag{448}$$

for some ζ between 0 and t, where (448) follows from (441). Given a compact set S in $\mathbb{R}$ there is a $\rho > 0$ such that $S \subseteq [-\rho, \rho]$, and thus it is sufficient to prove uniform convergence on $[-\rho, \rho]$. For $t \in [-\rho, \rho]$, from the triangle inequality one has

$$\begin{aligned} |f(t) - g(t)| &\le |f(t) - P_{6N+5}[f](t)| + |P_{6N+5}[f](t) \\ &\quad - P_{6N+5}[g](t)| + |P_{6N+5}[g](t) - g(t)| \\ &= |R_{6N+5}[f](t)| + |P_{6N+5}[f](t) \\ &\quad - P_{6N+5}[g](t)| + |R_{6N+5}[g](t)|. \end{aligned} \tag{449}$$

Now for $t \in [-\rho, \rho]$ and relying on (448), one sees

$$\begin{aligned} |R_{6N+5}[f](t)| &= \left| \frac{(-1)^{N+1} q^{(N+1)(N+1-4/3)/(1/3)} f(q^{N+1}\zeta)}{(6N+6)!} t^{6N+6} \right| \\ &\le \frac{q^{(N+1)(N+1-4/3)/(1/3)} \rho^{6N+6}}{(6N+6)!} \left| f(q^{N+1}\zeta) \right| \end{aligned} \tag{450}$$

$$\leq \frac{q^{(N+1)(N+1-4/3)/(1/3)}\rho^{6N+6}}{(6N+6)!}[1], \tag{451}$$

where moving from (450) to (451) is obtained from the global bound (424). Similarly, from (447), one has

$$|R_{6N+5}[g](t)| = \left|\frac{(-1)^{N+1}g(\xi)}{(6N+6)!}t^{6N+6}\right| \leq \frac{\rho^{6N+6}}{(6N+6)!}, \tag{452}$$

as $|g(\xi)| = |\cos(\xi)| \leq 1$. Also, from (443) and (444)–(446) and for $N \geq 2$, we have

$$\begin{aligned}
|P_{6N+5}[f](t) - P_{6N+5}[g](t)| &= \left|\sum_{k=0}^{N}\frac{(-1)^k\{q^{3k(k-1)}q^{(-1)(k-1)-1}-1\}}{(6k)!}t^{6k} + \sum_{k=0}^{N}\frac{(-1)^{k+1}\{q^{3k(k-1)}q^{(k-1)}-1\}}{(6k+2)!}t^{6k+2}\right.\\
&\quad\left.+\sum_{k=0}^{N}\frac{(-1)^k\{q^{3k(k-1)}q^{3(k-1)-1}\theta(q^6;-q^3)/[-2\theta(q^6;-q^{-1})]-1\}}{(6k+4)!}t^{6k+4}\right|\\
&\leq \sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{(-1)(k-1)-1}-1|}{(6k)!}\rho^{6k} + \sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{(k-1)}-1|}{(6k+2)!}\rho^{6k+2}\\
&\quad+\sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{3(k-1)-1}\theta(q^6;-q^3)/[-2\theta(q^6;-q^{-1})]-1|}{(6k+4)!}\rho^{6k+4}\\
&\leq \sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{(-1)(k-1)-1}-1|}{(6k)!}\rho^{6k} + \sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{(k-1)}-1|}{(6k+2)!}\rho^{6k+2}\\
&\quad+\sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{3(k-1)-1}(\theta(q^6;-q^3)/[-2\theta(q^6;-q^{-1})])-q^{3k(k-1)}q^{3(k-1)-1}|}{(6k+4)!}\rho^{6k+4}\\
&\quad+\sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{3(k-1)-1}-1|}{(6k+4)!}\rho^{6k+4},
\end{aligned} \tag{453}$$

which, after rearranging and factoring, equals

$$\begin{aligned}
&\sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{(-1)(k-1)-1}-1|}{(6k)!}\rho^{6k} + \sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{(k-1)}-1|}{(6k+2)!}\rho^{6k+2}\\
&\quad+\sum_{k=0}^{N}\frac{|q^{3k(k-1)}q^{3(k-1)-1}-1|}{(6k+4)!}\rho^{6k+4} + \left|\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]}-1\right|\\
&\quad\cdot\sum_{k=0}^{N}\frac{q^{3k(k-1)}q^{3(k-1)-1}}{(6k+4)!}\rho^{6k+4} \leq \left[q^{3N(N-1)}q^{3(N-1)-1}-1\right]\\
&\quad\cdot\left[\sum_{k=0}^{N}\frac{\rho^{6k}}{(6k)!}+\sum_{k=0}^{N}\frac{\rho^{6k+2}}{(6k+2)!}+\sum_{k=0}^{N}\frac{\rho^{6k+4}}{(6k+4)!}\right]\\
&\quad+\left|\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]}-1\right|q^{3N(N-1)}q^{3(N-1)-1}\sum_{k=0}^{N}\frac{\rho^{6k+4}}{(6k+4)!}\\
&\leq \left[q^{3N(N-1)}q^{3(N-1)-1}-1\right][e^{\rho}] + \left|\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]}-1\right|\\
&\quad\cdot q^{3N(N-1)}q^{3(N-1)-1}e^{\rho}.
\end{aligned} \tag{454}$$

Applying (451), (454), and (452), respectively, to the terms in (449), one has that for each $N \geq 2$

$$\begin{aligned}
|f(t)-g(t)| &\leq q^{(N+1)(N+1-4/3)/(1/3)}\frac{\rho^{6N+6}}{(6N+6)!}[1]\\
&\quad+\left[q^{3N(N-1)}q^{3(N-1)-1}-1\right][e^{\rho}]
\end{aligned} \tag{455}$$

$$+\left|\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]}-1\right|q^{3N(N-1)}q^{3(N-1)-1}e^{\rho} + \frac{\rho^{6N+6}}{(6N+6)!}. \tag{456}$$

Given $\varepsilon > 0$, choose $N_0 \geq 2$ sufficiently large such that one has $\rho^{6N_0+6}/[(6N_0+6)!] < \varepsilon/4$ to begin bounding the second term in (456). To handle the terms in (455), note that

$$\begin{gathered}
1 < (\varepsilon/4)[(6N_0+6)!]/[\rho^{6N_0+6}] \text{ and,}\\
\text{automatically, } 1 < 1+\varepsilon/[4e^{\rho}],
\end{gathered} \tag{457}$$

and pick $q_0 > 1$ so that

$$q_0^{(N_0+1)(N_0+1-4/3)/(1/3)} < \frac{(\varepsilon/4)[(6N_0+6)!]}{[\rho^{6N_0+6}]}, \tag{458}$$

$$q_0^{3N_0(N_0-1)}q_0^{3(N_0-1)-1} < 1+\frac{\varepsilon}{[4e^{\rho}]} \tag{459}$$

Then, for $1 < q < q_0$, one has

$$q^{(N_0+1)(N_0+1-4/3)/(1/3)} < \frac{(\varepsilon/4)[(6N_0+6)!]}{[\rho^{6N_0+6}]}, \quad (460)$$

$$q^{3N_0(N_0-1)}q^{3(N_0-1)-1} < 1 + \frac{\varepsilon}{[4e^\rho]}. \quad (461)$$

Hence, for $1 < q < q_0$

$$\frac{\rho^{6N_0+6}}{(6N_0+6)!} < q^{(N_0+1)(N_0+1-4/3)/(1/3)}\frac{\rho^{6N_0+6}}{(6N_0+6)!} < \varepsilon/4, \quad (462)$$

and

$$\left[q^{3N_0(N_0-1)}q^{3(N_0-1)-1} - 1\right]e^\rho < \varepsilon/4, \quad (463)$$

where (462)–(463) control the terms in (455) as well as the last term in (456). To handle the remaining term in (456), pick q_1 with $1 < q_1 < q_0$ such that for all $1 < q < q_1$ one has

$$\left|\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]} - 1\right| < \varepsilon/\left[4q_0^{3N_0(N_0-1)}q_0^{3(N_0-1)-1}e^\rho\right], \quad (464)$$

which follows from the fact that

$$\lim_{q\longrightarrow 1^+}\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]} = 1, \quad (465)$$

via (440) above and/or the generalized q-Wallis formula (482) below (in the case that $m = 6, K = -1$, and $J = 3$). Then for $1 < q < q_1$, it follows that

$$\begin{aligned}&\left|\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]} - 1\right|q^{3N_0(N_0-1)}q^{3(N_0-1)-1}e^\rho \\ &\quad < \left|\frac{\theta(q^6;-q^3)}{[-2\theta(q^6;-q^{-1})]} - 1\right|q_0^{3N_0(N_0-1)}q_0^{3(N_0-1)-1}e^\rho < \frac{\varepsilon}{4}.\end{aligned} \quad (466)$$

Applying (462) and (463) along with (466) to (455) and (456) with N taken to be N_0 one has that for $1 < q < q_1$

$$|f(t) - g(t)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \quad (467)$$

for $t \in [-\rho, \rho]$. So $f(t) = \mathscr{W}_{-4/3,1/3}(t)/\mathscr{W}_{-4/3,1/3}(0)$ approaches $g(t) = \cos(t)$ uniformly on $[-\rho, \rho]$ as $q \longrightarrow 1^+$, and the proposition is proven. □

7.3. Generalized q-Wallis Formulas. In [6], we have proven a q-Wallis formula given by the first equality in (468)

$$\begin{aligned}\lim_{q\longrightarrow 1^+}\frac{\ln(q)\left(\mu_{q^2}\right)^3}{\theta(q^2,-1/q)} &= \frac{\pi}{2} = \prod_{n=1}^{\infty}\left[\frac{(2n)^2}{(2n-1)(2n+1)}\right] \\ &= \left[\frac{2}{1}\frac{2}{3}\right]\cdot\left[\frac{4}{3}\frac{4}{5}\right]\cdot\left[\frac{6}{5}\frac{6}{7}\right]\cdots,\end{aligned} \quad (468)$$

where the last two equalities of (468) are Wallis' formula for $\pi/2$. We now finish the paper by generalizing the above result in order to provide a number of related generalized q-Wallis formulas.

Theorem 46. *Let $m \in \mathbb{N}$ with $m \geq 2$, and let $j \in \{0, 1, \cdots, m-1\}$ with $k \in \{1, 2, \cdots, m-1\}$. Then the following families of generalized q-Wallis formulas hold:*

$$\lim_{q\longrightarrow 1^+}\frac{\ln(q)\left(\mu_{q^m}\right)^3}{\theta(q^m,-q^k)} = \frac{-\pi/m}{\sin(k\pi/m)}, \quad (469)$$

$$\lim_{q\longrightarrow 1^+}\frac{\theta(q^m,-q^j)}{\theta(q^m,-q^k)} = \frac{\sin(j\pi/m)}{\sin(k\pi/m)}. \quad (470)$$

Proof. The proof relies on the following factorization of $\sin(x)/x$:

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right). \quad (471)$$

From (10) and (11), one has

$$\begin{aligned}&\frac{\ln(q)\left(\mu_{q^m}\right)^3}{\theta(q^m,-q^k)} \\ &= \frac{\ln(q)\,\mu_{q^m}\prod_{n\geq1}[(1-1/q^{mn})(1-1/q^{mn})]}{\mu_{q^m}\prod_{n\geq0}\left[\left(1-q^k/q^{mn}\right)\left(1-1/\left[q^k q^{m(n+1)}\right]\right)\right]} \\ &= \frac{\ln(q)\prod_{n\geq1}[(1-1/q^{mn})(1-1/q^{mn})]}{\left(1-q^k\right)\prod_{n\geq0}\left[\left(1-q^k/q^{m(n+1)}\right)\left(1-1/\left[q^k q^{m(n+1)}\right]\right)\right]} \\ &= \frac{\ln(q)\prod_{n\geq1}[(1-1/q^{mn})(1-1/q^{mn})]}{(-1)(q-1)\left(\sum_{\ell=0}^{k-1}q^\ell\right)\prod_{n\geq1}\left[\left(1-1/q^{mn-k}\right)\left(1-1/q^{mn+k}\right)\right]} \\ &= \frac{-\ln(q)}{(q-1)\left(\sum_{\ell=0}^{k-1}q^\ell\right)}\prod_{n\geq1}\left[\frac{\left(1/q^{2mn}\right)(q^{mn}-1)(q^{mn}-1)}{\left(1/q^{2mn}\right)\left(q^{mn-k}-1\right)\left(q^{mn+k}-1\right)}\right] \\ &= \frac{-\ln(q)}{(q-1)\left(\sum_{\ell=0}^{k-1}q^\ell\right)}\prod_{n\geq1}\left[\frac{(q-1)^2\left(\sum_{\ell=0}^{mn-1}q^\ell\right)\left(\sum_{\ell=0}^{mn-1}q^\ell\right)}{(q-1)^2\left(\sum_{\ell=0}^{mn-k-1}q^\ell\right)\left(\sum_{\ell=0}^{mn+k-1}q^\ell\right)}\right] \\ &= \frac{-\ln(q)}{(q-1)\left(\sum_{\ell=0}^{k-1}q^\ell\right)}\prod_{n\geq1}\left[\frac{\left(\sum_{\ell=0}^{mn-1}q^\ell\right)\left(\sum_{\ell=0}^{mn-1}q^\ell\right)}{\left(\sum_{\ell=0}^{mn-k-1}q^\ell\right)\left(\sum_{\ell=0}^{mn+k-1}q^\ell\right)}\right],\end{aligned} \quad (472)$$

where we have reindexed replacing $n+1$ by n in the denominator to obtain (472). Relying on the facts that $[\lim_{q\longrightarrow 1^+} \ln(q)/(q-1)] = 1$ and $[\lim_{q\longrightarrow 1^+} \sum_{\ell=0}^{p} q^\ell] = \sum_{\ell=0}^{p} 1 = p+1$, one sees (since $k \in \{1, 2, \cdots, m-1\}$) that

$$\lim_{q\longrightarrow 1^+} \frac{\ln(q)\left(\mu_{q^m}\right)^3}{\theta(q^m, -q^k)} = \lim_{q\longrightarrow 1^+} \frac{-\ln(q)}{(q-1)\left(\sum_{\ell=0}^{k-1} q^\ell\right)} \prod_{n\geq 1} \left[\frac{\left(\sum_{\ell=0}^{mn-1} q^\ell\right)\left(\sum_{\ell=0}^{mn-1} q^\ell\right)}{\left(\sum_{\ell=0}^{mn-k-1} q^\ell\right)\left(\sum_{\ell=0}^{mn+k-1} q^\ell\right)}\right] \tag{473}$$

$$= \frac{-1}{k} \prod_{n\geq 1} \frac{(mn)^2}{(mn-k)(mn+k)} = \frac{-1}{k} \prod_{n\geq 1} \frac{(mn)^2}{(mn)^2 - k^2} \tag{474}$$

$$= \frac{-1}{k} \prod_{n\geq 1} \frac{1}{1-(k/m)^2/n^2} = \frac{-1}{k} \frac{1}{\prod_{n\geq 1}\left[1-(k\pi/m)^2/\pi^2 n^2\right]} \tag{475}$$

$$= \frac{-1}{k} \frac{k\pi/m}{\sin(k\pi/m)} = \frac{-\pi/m}{\sin(k\pi/m)}, \tag{476}$$

giving (469), where (471) was used to obtain the first equality in (476). Now when $j=0$ (470) holds since $\theta(q^m; -1) = 0$ (which in turn follows from (13)). For $j \in \{1, 2, \cdots, m-1\}$, one now has from (469) that

$$\lim_{q\longrightarrow 1^+} \frac{\theta(q^m, -q^j)}{\theta(q^m, -q^k)} = \lim_{q\longrightarrow 1^+} \frac{\theta(q^m, -q^j)}{\ln(q)\left(\mu_{q^m}\right)^3} \frac{\ln(q)\left(\mu_{q^m}\right)^3}{\theta(q^m, -q^k)} = \frac{\sin(j\pi/m)}{(-\pi/m)} \frac{(-\pi/m)}{\sin(k\pi/m)} = \frac{\sin(j\pi/m)}{\sin(k\pi/m)}, \tag{477}$$

giving (470). This finishes the proof of the theorem. □

Remark 47. We call the above results generalized q-Wallis formulas via the following reasoning. First, note that the left-most infinite product

$$\prod_{n\geq 1} \frac{(mn)^2}{(mn-k)(mn+k)} \tag{478}$$

in (474) generalizes the infinite product

$$\prod_{n=1}^{\infty} \left[\frac{(2n)^2}{(2n-1)(2n+1)}\right], \tag{479}$$

in the Wallis formula for $\pi/2$ in (468). In particular note that when $m=2$ and $k=1$ the infinite product in (478) becomes the infinite product in (479), and the Wallis product for $\pi/2$ is then duplicated in (474)–(476). Second, note that the product

$$\prod_{n\geq 1} \left[\frac{\left(\sum_{\ell=0}^{mn-1} q^\ell\right)\left(\sum_{\ell=0}^{mn-1} q^\ell\right)}{\left(\sum_{\ell=0}^{mn-k-1} q^\ell\right)\left(\sum_{\ell=0}^{mn+k-1} q^\ell\right)}\right] \tag{480}$$

in (473) is the q-analogue of the product (478). Hence, we are introducing generalized q-Wallis formulas.

Corollary 48. *Let $m \in \mathbb{N}$ satisfy $m \geq 2$. Let $J = mL + j \in \mathbb{Z}$ with $j \in \{0, 1, \cdots, m-1\}$ and $K = m\ell + k \in \mathbb{Z}$ with $k \in \{1, \cdots, m-1\}$ then*

$$\lim_{q\longrightarrow 1^+} \frac{\ln(q)\left(\mu_{q^m}\right)^3}{\theta(q^m, -q^K)} = \frac{(-1)^{\ell+1}\pi/m}{\sin(k\pi/m)}, \tag{481}$$

$$\lim_{q\longrightarrow 1^+} \frac{\theta(q^m, -q^J)}{\theta(q^m, -q^K)} = (-1)^{L+\ell} \frac{\sin(j\pi/m)}{\sin(k\pi/m)}. \tag{482}$$

Proof. Relying on (12), one has

$$\lim_{q\longrightarrow 1^+} \frac{\ln(q)\left(\mu_{q^m}\right)^3}{\theta(q^m, -q^K)} = \lim_{q\longrightarrow 1^+} \frac{\ln(q)\left(\mu_{q^m}\right)^3}{\theta(q^m, -q^{m\ell+k})} = \lim_{q\longrightarrow 1^+} \frac{\ln(q)\left(\mu_{q^m}\right)^3}{(q^m)^{\ell(\ell+1)/2}(-q^k)^\ell \theta(q^m, -q^k)} = (-1)^\ell \frac{(-\pi/m)}{\sin k\pi/m}, \tag{483}$$

giving (481), where the last equality follows from (469). If $j=0$ then (482) holds since then $\theta(q^m; -q^{mL}) = 0$ by (13). If on the other hand $j \in \{1, 2, \cdots, m-1\}$ then from (481)

$$\lim_{q\longrightarrow 1^+} \frac{\theta(q^m, -q^J)}{\theta(q^m, -q^K)} = \lim_{q\longrightarrow 1^+} \frac{\theta(q^m, -q^J)}{\ln(q)\left(\mu_q^m\right)^3} \frac{\ln(q)\left(\mu_q^m\right)^3}{\theta(q^m, -q^K)} = \frac{\sin(j\pi/m)}{(-1)^{L+1}\pi/m} \frac{(-1)^{\ell+1}\pi/m}{\sin(k\pi/m)} = (-1)^{L+\ell} \frac{\sin(j\pi/m)}{\sin(k\pi/m)}, \tag{484}$$

giving (482). Observe that when $m=2$ and $K=-1=2(-1)+1=m\ell+j$, then setting $m=2$, $K=-1$, $\ell=-1$, and $k=1$ in (481) recovers the q-Wallis limit in (468). This proves the corollary. □

We point out that in the case of rational exponents $\lim_{q\longrightarrow 1^+} \theta(q^{m/n}; -q^{a/b})/\theta(q^{m/n}; -q^{c/d})$ can be evaluated by obtaining a common denominator N for the fractions in the exponents and reexpressing them as $m/n = M/N$, $a/b = A/N$, and $c/d = C/N$ to obtain:

Corollary 49. *In the case of rational exponents*

$$\lim_{q\longrightarrow 1^+}\frac{\theta\left(q^{m/n};-q^{a/b}\right)}{\theta\left(q^{m/n};-q^{c/d}\right)}=\lim_{q\longrightarrow 1^+}\frac{\theta\left(q^{M/N};-q^{A/N}\right)}{\theta\left(q^{M/N};-q^{C/N}\right)}=\lim_{Q\longrightarrow 1^+}\frac{\theta\left(Q^{M};-Q^{A}\right)}{\theta\left(Q^{M};-Q^{C}\right)}=(-1)^{L+\ell}\frac{\sin\left(j\pi/M\right)}{\sin\left(k\pi/M\right)}, \tag{485}$$

where $Q=q^{1/N}$ *and* $A=ML+j$ *and* $C=M\ell+k$ *with* $j,k\in\{0,1,\cdots,M-1\}$ *and* $k\neq 0$.

Proof. One applies Corollary 48 with $A=ML+j$ and $C=M\ell+k$ with $j,k\in\{0,1,\cdots,M-1\}$ and $k\neq 0$ to obtain the last equation in (485). This proves the corollary. □

We point out that, in our setting, important and useful applications of these generalized q-Wallis formulas (especially those of type (470), (482), and (485)) occur in proving

(1) initial conditions for a classical ODE analogous to a given MADE as seen earlier in (439) and (440)

(2) convergence of MADE solutions to their classical analogous ODE as seen in (464) and (465) in Proposition 45

Data Availability

There are no further data beyond that submitted in the paper.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

Acknowledgments

This work was supported by the ECU Thomas Harriot College of Arts and Sciences, the Open Access Publishing Support Fund sponsored by ECU Libraries, and the ECU Mathematics Department.

References

[1] G. H. Hardy and N. Riesz, *The General Theory of Dirichlet's Series*, Cambridge University Press, Cambridge, 1915.

[2] D. Pravica, N. Randriampiry, and M. Spurr, "Solutions of a class of multiplicatively advanced differential equations," *Comptes Rendus Mathematique*, vol. 356, no. 7, pp. 776–817, 2018.

[3] D. Pravica, N. Randriampiry, and M. Spurr, "Applications of an advanced differential equation in the study of wavelets," *Applied and Computational Harmonic Analysis*, vol. 27, no. 1, pp. 2–11, 2009.

[4] D. Pravica, N. Randriampiry, and M. Spurr, "Reproducing kernel bounds for an advanced wavelet frame via the theta function," *Applied and Computational Harmonic Analysis*, vol. 33, no. 1, pp. 79–108, 2012.

[5] D. Pravica, N. Randriampiry, and M. Spurr, "Theta function identities in the study of wavelets satisfying advanced differential equations," *Applied and Computational Harmonic Analysis*, vol. 29, no. 2, pp. 134–155, 2010.

[6] D. Pravica, N. Randriampiry, and M. Spurr, "On q -advanced spherical Bessel functions of the first kind and perturbations of the Haar wavelet," *Applied and Computational Harmonic Analysis*, vol. 44, no. 2, pp. 350–413, 2018.

[7] L. Fox, D. F. Mayers, J. R. Ockendon, and A. B. Tayler, "On a functional differential equation," *IMA Journal of Applied Mathematics*, vol. 8, no. 3, pp. 271–307, 1971.

[8] T. Kato and J. B. McLeod, "The functional-differential equation $y'(x)=ay(\lambda x)+by(x)$," *Bulletin of the American Mathematical Society*, vol. 77, no. 6, pp. 891–937, 1971.

[9] N. T. Dung, "Asymptotic behavior of linear advanced differential equations," *Acta Mathematica Scientia*, vol. 35, no. 3, pp. 610–618, 2015.

[10] L. Di Vizio, "An ultrametric version of the Maillet-Malgrange theorem for nonlinear q-difference equations," *Proceedings of the American Mathematical Society*, vol. 136, no. 8, pp. 2803–2814, 2008.

[11] L. Di Vizio and C. Hardouin, "Descent for differential Galois theory of difference equations: confluence andq-dependence," *Pacific Journal of Mathematics*, vol. 256, no. 1, pp. 79–104, 2012.

[12] L. Di Vizio and C. Zhang, "On q-summation and confluence," *Annales de l'institut Fourier*, vol. 59, no. 1, pp. 347–392, 2009.

[13] T. Dreyfus, "Building meromorphic solutions of q-difference equations using a Borel-Laplace summation," *International Mathematics Research Notices*, vol. 15, pp. 6562–6587, 2015.

[14] T. Dreyfus, A. Lastra, and S. Malek, "On the multiple-scale analysis for some linear partial q-difference and differential equations with holomorphic coefficients," *Advances in Differential Equations*, vol. 326, pp. 1–42, 2019.

[15] A. Lastra and S. Malek, "On q-Gevrey asymptotics for singularly perturbed q-difference-differential problems with an irregular singularity," *Abstract and Applied Analysis*, vol. 2012, Article ID 860716, 35 pages, 2012.

[16] A. Lastra and S. Malek, "On parametric Gevrey asymptotics for singularly perturbed partial differential equations with delays," *Abstract and Applied Analysis*, vol. 2013, Article ID 723040, 18 pages, 2013.

[17] A. Lastra and S. Malek, "Parametric Gevrey asymptotics for some Cauchy problems in quasiperiodic function spaces," *Abstract and Applied Analysis*, vol. 2014, Article ID 153169, 29 pages, 2014.

[18] A. Lastra and S. Malek, "On parametric multilevel q-Gevrey asymptotics for some linear q-difference-differential equations," *Adv. Difference Equ.*, vol. 344, 2015.

[19] A. Lastra and S. Malek, "On multiscale Gevrey and q-Gevrey asymptotics for some linear q-difference differential initial value Cauchy problems," *Journal of Difference Equations and Applications*, vol. 23, no. 8, pp. 1397–1457, 2017.

[20] A. Lastra and S. Malek, "On a q-analog of a singularly perturbed problem of irregular type with two complex time variables," *Mathematics*, vol. 7, no. 10, pp. 924-925, 2019.

[21] A. Lastra, S. Malek, and J. Sanz, "On q-asymptotics for linear q-difference-differential equations with Fuchsian and

irreguular singularities," *Journal of Differential Equations*, vol. 252, no. 10, pp. 5185–5216, 2012.

[22] A. Lastra, S. Malek, and J. Sanz, "On q-asymptotics for q-difference-differential equations with Fuchsian and irregular singularities, formal and analytic solutions of differential and difference equations," *Polish Acad. Sci. Inst. Math.*, vol. 97, 2012.

[23] A. Lastra, S. Malek, and J. Sanz, "On Gevrey solutions of threefold singular nonlinear partial differential equations," *Journal of Differential Equations*, vol. 255, no. 10, pp. 3205–3232, 2013.

[24] S. Malek, "On complex singularity analysis for linear partial q -difference-differential equations using nonlinear differential equations," *Journal of Dynamical and Control Systems*, vol. 19, no. 1, pp. 69–93, 2013.

[25] S. Malek, "On parametric Gevrey asymptotics for a q-analog of some linear initial value problem," *Fako de l'Funkcialaj Ekvacioj Japana Matematika Societo*, vol. 60, no. 1, pp. 21–63, 2017.

[26] S. Malek, "On a partial q-analog of a singularly perturbed problem with Fuchsian and irregular time ingularities," *Abstract and Applied Analysis*, vol. 2020, Article ID 7985298, 32 pages, 2020.

[27] S. Malek, "Asymptotics and confluence for some linear q-difference-differential Cauchy problem," *The Journal of Geometric Analysis*, vol. 32, no. 93, 2022.

[28] H. Tahara, "Q-analogues of Laplace and Borel transforms by means of q-exponentials," *Annales de l'institut Fourier*, vol. 67, no. 5, pp. 1865–1903, 2017.

[29] C. Zhang, "Analytic continuation of solutions of the pantograph equation by means of θ-modular forms," 2012, http://arxiv.org/abs/1202.0423.

[30] D. Pravica and M. Spurr, "Analytic continuation int the future," *Discrete and Continuous Dynamical Systems, Supplement*, vol. 2002, pp. 709–716, 2002.

[31] D. Pravica and M. Spurr, "Unique summing of formal power seris solutions to advanced and delayed differential equations," *Discrete and Continuous Dynamical Systems, Supplement*, vol. 2005, pp. 730–737, 2005.

[32] D. Pravica, N. Randriampiry, and M. Spurr, "q-advanced models for tsunami and rogue waves," *Abstract and Applied Analysis*, vol. 2012, Article ID 414060, 26 pages, 2012.

[33] D. Pravica, N. Randriampiry, and M. Spurr, "Smooth wavelet approximations of truncated Legendre polynomials via the Jacobi theta function," *Abstract and Applied Analysis*, vol. 2014, Article ID 890456, 24 pages, 2014.

[34] D. Pravica, N. Randriampiry, and M. Spurr, "Eigenfunction families and solution bounds for multiplicatively advanced differential equations," *Axioms*, vol. 9, no. 3, p. 83, 2020.

[35] S. Krantz, *Handbook of Complex Variables*, Birkhäuser, 1999.

[36] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[37] I. Daubechies, *Ten Lectures on Wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 61, SIAM, Philadelphia, 1992.