Journal of Advances in Mathematics and Computer Science

26(1): 1-11, 2018; Article no.JAMCS.34424

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

A Class of Linear Multi-step Method for Direct Solution of Second Order Initial Value Problems in Ordinary Differential Equations by Collocation Method

N. S. Yakusak[1*] and A. O. Owolanke[2]

[1]*Department of Mathematics, Federal College of Education, Okene, Kogi State, Nigeria.*
[2]*Department of Mathematics, Ondo State University of Science and Technology, Okitipupa, Nigeria.*

Authors' contributions

This work was carried out in collaboration between both authors. Author NSY proposed, derived and implements the method. Author AOO analyze and present the numerical results graphical. Both authors managed the literature searches, read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/34424
Editor(s):
(1) Jacek Dziok, Professor, Institute of Mathematics, University of Rzeszow, Poland.
(2) Paul Bracken, Professor, Department of Mathematics, The University of Texas-Pan American, Edinburg, USA.
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(5) Jaywan Chung, Korea Electrotechnology Research Institute, Republic of Korea.
Complete Peer review History: http://www.sciencedomain.org/review-history/22741

Original Research Article

Received: 26th May 2017
Accepted: 30th August 2017
Published: 15th January 2018

Abstract

In this work, we proposed a linear multi-step method for solution of second order Initial Value Problems (IVP), using power series function as the trail solution for the approximation via collocation techniques. The resulting scheme is self-starting, consistent, zero-stable, convergent with good region of absolute stability. Numerical and graphical results are presented tabularly.

Keywords: Block methods; convergent; collocation; initial value problem.

**Corresponding author: E-mail: yakusaksamuel@yahoo.com;*

1 Introduction

Consider the second order initial value problems in ordinary differential equation of the form

$$y'' = f(x,\ y, y'), y(x_0) = y_0, y'(x_0) = y'_0 \tag{1}$$

Where f is continuous and satisfies Lipschitz condition. The solution of (1) is applicable in areas such as: Models of Chemical reaction, deflection and deformation of beam, heat transmission among others. Many authors such as [1-5,6] among others, extensively contributes for the solution of (1) without reducing it to system of first order initial value problems. Recently, [7] derived a new hybrid block method of order five with three off-step points for solving second order ODE directly via collocation and interpolation techniques. [8]. Make used of Legendre Polynomial as the basis function of the approximation for direct solution of initial value problems. In this paper, we make us of power series as the basis function for the approximation, which simultaneously generate numerical result for the solution of (1), without reducing to system of first order ODEs.

2 Derivation of the Method

In this section, our objective is to derived a Linear Multi-step Method (LMM) in the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2[\sum_{j=0}^{k}\beta_j f_{n+j} + \beta_v f_{n+v}] \tag{2}$$

where $\alpha = 1, \beta$ 6= 0 and α_0, β_0 are both not zero. α and β are real and continuous functions . In order to obtain (2), we seek an approximation $y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{3}$$

Where a_j are coefficients to be determine r and s are interpolating and collocating points. Imposing the following conditions

$$Y(x_{n+s}) = y_{n+s}, s = \frac{4}{6}, \frac{5}{6} \tag{4}$$

$$Y''(x_{n+r}) = f_{n+r}, r = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1 \tag{5}$$

From equation (4) and (5) the system of $k + 9$ equations and $k + 9$ unknowns is obtained. By Gaussian elimination method the coefficients of a_j are obtained, and then substituting the values of a_j into equation (3), yield the continuous method below

$$Y(x)=\sum_{j=0}^{k}\alpha_j(x)y_{n+j}+\sum_{v_r}\alpha_{v_r}(x)y_{n+v_r}+h^2\sum_{j=0}^{k}\beta_k(x)f_{n+k}+h^2\sum_{v_r}\beta_{v_r}(x)f_{n+v_r}, r=\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6} \quad (6)$$

For $k=1,\ v_1=\frac{1}{6}, v_2=\frac{2}{6}, v_3=\frac{3}{6}, v_4=\frac{4}{6}, v_5=\frac{5}{6},\ j=0,\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},1$. We express $\alpha_j(x)$ and $\beta_j(x)$ as continuous coefficients, written as continuous function of *z*, by letting $z=\frac{x-x_n}{h}$, the following parameter are obtained as

$$\left.\begin{aligned}
&\alpha_{\frac{2}{3}}(z)=5-6z\\
&\alpha'_{\frac{5}{6}}(z)=-4+6z\\
&\beta_0(z)=h^2[1/2\,z^2-\frac{12287}{241920}\,z+\frac{81}{70}\,z^8-\frac{27}{5}\,z^7+21/2\,z^6-\frac{441}{40}\,z^5+\frac{203}{30}\,z^4-\frac{49}{20}\,z^3+\frac{409}{217728}]\\
&\beta_{\frac{1}{6}}(z)=h^2[-\frac{29369}{120960}\,z-\frac{243}{35}\,z^8+\frac{216}{7}\,z^7-\frac{279}{5}\,z^6+\frac{261}{5}\,z^5-\frac{261}{10}\,z^4+6\,z^3+\frac{1061}{36288}]\\
&\beta_{\frac{2}{6}}(z)=h^2[-\frac{2671}{34560}\,z+\frac{243}{14}\,z^8-\frac{513}{7}\,z^7+\frac{1233}{10}\,z^6-\frac{4149}{40}\,z^5+\frac{351}{8}\,z^4-15/2\,z^3+\frac{3893}{72576}]\\
&\beta_{\frac{3}{6}}(z)=h^2[-\frac{14719}{60480}\,z-\frac{162}{7}\,z^8+\frac{648}{7}\,z^7-\frac{726}{5}\,z^6+\frac{558}{5}\,z^5-\frac{127}{3}\,z^4+\frac{20}{3}\,z^3+\frac{4633}{54432}]\\
&\beta_{\frac{4}{6}}(z)=h^2[-\frac{5293}{48384}\,z+\frac{243}{14}\,z^8-\frac{459}{7}\,z^7+\frac{963}{10}\,z^6-\frac{2763}{40}\,z^5+\frac{99}{4}\,z^4-\frac{15}{4}\,z^3+\frac{7085}{72576}]\\
&\beta_{\frac{5}{6}}(z)=h^2[-\frac{3481}{120960}\,z-\frac{243}{35}\,z^8+\frac{864}{35}\,z^7-\frac{171}{5}\,z^6+\frac{117}{5}\,z^5-\frac{81}{10}\,z^4+6/5\,z^3+\frac{389}{36288}]\\
&\beta_1(z)=h^2[\frac{13}{5376}\,z+\frac{81}{70}\,z^8-\frac{27}{7}\,z^7+\frac{51}{10}\,z^6-\frac{27}{8}\,z^5+\frac{137}{120}\,z^4-1/6\,z^3-\frac{95}{217728}]
\end{aligned}\right\}$$

Solving (6) independently gives the continuous hybrid block method by imposing the condition $\delta_j(x)=\beta_0(x)$ written in the form

$$y'(x)=y'_n+h^2(\sum_{j=0}^{k}\delta_k(x)f_{n+k}+\delta_j(x)f_{n+j}), j=0,\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6} \quad (7)$$

where coefficient of f_{n+k} give

$$\left.\begin{aligned}
&\delta_0(z)=h[1/2\,z^2-\frac{27}{5}\,z^7+\frac{21}{2}\,z^6-\frac{441}{40}\,z^5+\frac{203}{30}\,z^4-\frac{49}{20}\,z^3+\frac{81}{70}\,z^8]\\
&\delta_{\frac{1}{6}}(z)=h[\frac{216}{7}\,z^7-\frac{279}{5}\,z^6+\frac{261}{5}\,z^5-\frac{261}{10}\,z^4+6\,z^3-\frac{243}{35}\,z^8]\\
&\delta_{\frac{2}{6}}(z)=h[-\frac{513}{7}\,z^7+\frac{1233}{10}\,z^6-\frac{4149}{40}\,z^5+\frac{351}{8}\,z^4-15/2\,z^3+\frac{243}{14}\,z^8]\\
&\delta_{\frac{3}{6}}(z)=h[\frac{648}{7}\,z^7-\frac{726}{5}\,z^6+\frac{558}{5}\,z^5-\frac{127}{3}\,z^4+\frac{20}{3}\,z^3-\frac{162}{7}\,z^8]\\
&\delta_{\frac{4}{6}}(z)=h[-\frac{459}{7}\,z^7+\frac{963}{10}\,z^6-\frac{2763}{40}\,z^5+\frac{99}{4}\,z^4-\frac{15}{4}\,z^3+\frac{243}{14}\,z^8]\\
&\delta_{\frac{5}{6}}(z)=h[\frac{864}{35}\,z^7-\frac{171}{5}\,z^6+\frac{117}{5}\,z^5-\frac{81}{10}\,z^4+6/5\,z^3-\frac{243}{35}\,z^8]\\
&\delta_1(z)=h[-\frac{27}{7}\,z^7+\frac{51}{10}\,z^6-\frac{27}{8}\,z^5+\frac{137}{120}\,z^4-1/6\,z^3+\frac{81}{70}\,z^8]
\end{aligned}\right\}$$

Evaluating (7) and its first derivatives at $\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6},1$ yields the discrete schemes below

$$\left.\begin{aligned}
y_{\frac{1}{6}} &= y_0 + \frac{1}{6}hy_0' + h^2\left(\frac{28549}{4354560}f_0 + \frac{275}{20736}f_{\frac{1}{6}} - \frac{5717}{483840}f_{\frac{1}{3}} + \frac{10621}{1088640}f_{\frac{1}{2}} - \frac{7703}{1451520}f_{\frac{2}{3}} + \frac{403}{241920}f_{\frac{5}{6}} - \frac{199}{870912}f_1\right)\\
y_{\frac{1}{3}} &= y_0 + \frac{1}{3}hy_0' + h^2\left(\frac{1027}{68040}f_0 + \frac{97}{1890}f_{\frac{1}{6}} - \frac{2}{81}f_{\frac{1}{3}} + \frac{197}{8505}f_{\frac{1}{2}} - \frac{97}{7560}f_{\frac{2}{3}} + \frac{23}{5670}f_{\frac{5}{6}} - \frac{19}{34020}f_1\right)\\
y_{\frac{1}{2}} &= y_0 + \frac{1}{2}hy_0' + h^2\left(\frac{253}{10752}f_0 + \frac{165}{1792}f_{\frac{1}{6}} - \frac{267}{17920}f_{\frac{1}{3}} + \frac{5}{128}f_{\frac{1}{2}} - \frac{363}{17920}f_{\frac{2}{3}} + \frac{57}{8960}f_{\frac{5}{6}} - \frac{47}{53760}f_1\right)\\
y_{\frac{2}{3}} &= y_0 + \frac{2}{3}hy_0' + h^2\left(\frac{272}{8505}f_0 + \frac{376}{2835}f_{\frac{1}{6}} - \frac{2}{945}f_{\frac{1}{3}} + \frac{656}{8505}f_{\frac{1}{2}} - \frac{2}{81}f_{\frac{2}{3}} + \frac{8}{945}f_{\frac{5}{6}} - \frac{2}{1701}f_1\right)\\
y_{\frac{5}{6}} &= y_0 + \frac{5}{6}hy_0' + h^2\left(\frac{35225}{870912}f_0 + \frac{8375}{48384}f_{\frac{1}{6}} + \frac{3125}{290304}f_{\frac{1}{3}} + \frac{25625}{217728}f_{\frac{1}{2}} - \frac{625}{96768}f_{\frac{2}{3}} + \frac{275}{20736}f_{\frac{5}{6}} - \frac{1375}{870912}f_1\right)\\
y_1 &= y_0 + hy_0' + h^2\left(\frac{41}{840}f_0 + \frac{3}{14}f_{\frac{1}{6}} + \frac{3}{140}f_{\frac{1}{3}} + \frac{17}{105}f_{\frac{1}{2}} + \frac{3}{280}f_{\frac{2}{3}} + \frac{3}{70}f_{\frac{5}{6}}\right)\\
y'_{\frac{1}{3}} &= y_0' + h\left(\frac{1139}{22680}f_0 + \frac{47}{189}f_{\frac{1}{6}} + \frac{47}{189}f_{\frac{1}{3}} + \frac{47}{189}f_{\frac{1}{2}} + \frac{11}{7560}f_{\frac{2}{3}} + \frac{166}{2835}f_{\frac{5}{6}} + \frac{11}{945}f_1\right)\\
y'_{\frac{1}{2}} &= y_0' + h\left(\frac{137}{2688}f_0 + \frac{27}{112}f_{\frac{1}{6}} + \frac{27}{112}f_{\frac{1}{3}} + \frac{27}{112}f_{1/2} + \frac{387}{4480}f_{\frac{2}{3}} + \frac{17}{105}f_{\frac{5}{6}} + \frac{9}{560}f_1\right)\\
y'_{\frac{2}{3}} &= y_0' + h\left(\frac{143}{2835}f_0 + \frac{232}{945}f_{\frac{1}{6}} + \frac{232}{945}f_{\frac{1}{3}} + \frac{232}{945}f_{\frac{1}{2}} + \frac{64}{945}f_{\frac{2}{3}} + \frac{752}{2835}f_{\frac{5}{6}} + \frac{8}{945}f_1\right)\\
y'_{\frac{5}{6}} &= y_0' + h\left(\frac{3715}{72576}f_0 + \frac{725}{3024}f_{\frac{1}{6}} + \frac{725}{3024}f_{\frac{1}{3}} + \frac{725}{3024}f_{\frac{1}{2}} + \frac{2125}{24192}f_{\frac{2}{3}} + \frac{125}{567}f_{\frac{5}{6}} + \frac{235}{3024}f_1\right)\\
y'_1 &= y_0' + h\left(\frac{41}{840}f_0 + \frac{9}{35}f_{\frac{1}{6}} + \frac{9}{35}f_{\frac{1}{3}} + \frac{9}{35}f_{\frac{1}{2}} + \frac{9}{280}f_{\frac{2}{3}} + \frac{34}{105}f_{\frac{5}{6}} + \frac{9}{35}f_1\right)
\end{aligned}\right\} \quad (8)$$

Equation (8) is the desired hybrid block method.

3 Analysis of the Method

In this section, the basic properties of the proposed method such as: order, error constants, consistency, zero-stability, convergence and region of absolute stability of the method will be discussed, to know the existence, order of accuracy, behavior and if the proposed method will give reasonable results.

3.1 Order and error constant of the proposed method

Let the linear difference operator L associated with the LMM Method (8) be defined as

$$\mathrm{L}(y(x)) = \sum_{j=0}^{k}\left[\alpha_j y(x_n + jh) - \beta_j hy'(x_n + jh) - h^2\beta_j y''(x_n + jh)\right] \quad (9)$$

where $y(x)$ is an arbitrary test function continuously differential on $[a, b]$. Expanding $y(x_n+jh)$, $y'(x_n + jh)$ and $y''(x_n + jh)$ of (8) in Taylor series in the form,

$$\left.\begin{aligned}
c_i &= \frac{m^i}{i!} + \frac{1}{i!}(\frac{1}{6})^i - \frac{1}{j!}\left(\frac{3}{14}\left(\frac{1}{6}\right)^j + \frac{3}{140}\left(\frac{2}{6}\right)^j + \frac{17}{105}\left(\frac{3}{6}\right)^j + \frac{3}{280}\left(\frac{4}{6}\right)^j + \frac{3}{70}\left(\frac{5}{6}\right)^j\right)\\
c_i &= \frac{m^i}{i!} + \frac{1}{i!}(\frac{2}{6})^i - \frac{1}{j!}\left(\frac{275}{20736}\left(\frac{1}{6}\right)^j - \frac{5717}{483840}\left(\frac{2}{6}\right)^j + \frac{10621}{1088640}\left(\frac{3}{6}\right)^j - \frac{7703}{1451520}\left(\frac{4}{6}\right)^j + \frac{403}{241920}\left(\frac{5}{6}\right)^j - \frac{199}{870912}(1)^j\right)\\
c_i &= \frac{m^i}{i!} + \frac{1}{i!}(\frac{3}{6})^i - \frac{1}{j!}\left(\frac{97}{1890}\left(\frac{1}{6}\right)^j - \frac{2}{81}\left(\frac{2}{6}\right)^j + \frac{197}{8505}\left(\frac{3}{6}\right)^j - \frac{97}{7560}\left(\frac{4}{6}\right)^j + \frac{23}{5670}\left(\frac{5}{6}\right)^j - \frac{19}{34020}(1)^j\right)\\
c_i &= \frac{m^i}{i!} + \frac{1}{i!}(\frac{4}{6})^i - \frac{1}{j!}\left(\frac{165}{1792}\left(\frac{1}{6}\right)^j - \frac{267}{17920}\left(\frac{2}{6}\right)^j + \frac{5}{128}\left(\frac{3}{6}\right)^j - \frac{363}{17920}\left(\frac{4}{6}\right)^j + \frac{57}{8960}\left(\frac{5}{6}\right)^j - \frac{47}{53760}(1)^j\right)\\
c_i &= \frac{m^i}{i!} + \frac{1}{i!}(\frac{5}{6})^i - \frac{1}{j!}\left(\frac{376}{2835}\left(\frac{1}{6}\right)^j - \frac{2}{945}\left(\frac{2}{6}\right)^j + \frac{656}{8505}\left(\frac{3}{6}\right)^j - \frac{2}{81}\left(\frac{4}{6}\right)^j + \frac{8}{945}\left(\frac{5}{6}\right)^j - \frac{2}{1701}(1)^j\right)\\
c_i &= \frac{m^i}{i!} + \frac{1}{i!}(1)^i - \frac{1}{j!}\left(\frac{8375}{48384}\left(\frac{1}{6}\right)^j + \frac{3125}{290304}\left(\frac{2}{6}\right)^j + \frac{25625}{217728}\left(\frac{3}{6}\right)^j - \frac{625}{96768}\left(\frac{4}{6}\right)^j + \frac{275}{20736}\left(\frac{5}{6}\right)^j - \frac{1375}{870912}(1)^j\right)
\end{aligned}\right\} \quad (10)$$

Where $m = \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1, j = 1...7, i = 0...9$. If we assume that $y(x)$ has many higher derivatives and collecting the terms we have.

$$L(y(x):h) = C_0 y(x) + C_1 h y^0(x) + C_2 h^2 y^{00}(x) + ... + C_p h^p y^p(x) + C_{p+2} h^{p+2} y^p(x) \tag{11}$$

According to Henrici [9], equation (10) has order p if $C_0 = C_1 = ...C_P = C_{p+1} = 0, C_{p+2}$ 6= 0.

The proposed method is of order $p = (7,7,7,7,7,7)$ with error constant

$$C_9 = \left(\frac{1}{195955200}, \frac{6031}{9142485811200}, \frac{233}{142851340800}, \frac{1}{391910400}, \frac{31}{8928208800}, \frac{1625}{365699432448}\right)^T$$

3.2 Zero-stability and consistency of the proposed method

The block method (8) is said to be zero-stable as $h \to 0$, if the roots of the first characteristics polynomial defined by

$$\rho(r) = det[rA^0 - A^0] \tag{12}$$

satisfies $|r_s| \leq 1$ and every root of $|r_s| = 1$ has multiplicity not exceeding the order of the differential equation. (see [10]).

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We have $\rho(r) = r^5(r - 1) = 0, r = 0,0,0,0,0,1$. The sufficient condition for linear k-step method (8) to be consistent if it has order $p \geq 1$ [11]. Thus, the proposed method is consistent.

3.3 Convergence of the proposed method

The two sufficient conditions for a linear hybrid multi-step methods to be convergent is for it to be zero-stable and consistent [12]. Our proposed method converges since it satisfies the two conditions.

3.4 Region of absolute stability of the proposed method

Definition 1. A linear multi-step method is said to be A-stable if its region of absolute stability, contains the whole of the left-hand complex half-plane $R(h\lambda) < 0$. [13] It is important to investigate the performance of the method in the case of $h > 0$ fixed. We formulated the stability matrix as follows

$$M(z) = \eta(A - Cz - Dz^2) - B \tag{13}$$

where $z = \lambda h$, A, B, C, D are obtained from interpolating and collocating points of the method as

$$A=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},B=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},C=\begin{bmatrix}0&0&0&0&0&-\frac{1}{6}\\0&0&0&0&0&-\frac{3}{6}\\0&0&0&0&0&-\frac{2}{6}\\0&0&0&0&0&-\frac{4}{6}\\0&0&0&0&0&-\frac{5}{6}\\0&0&0&0&0&-1\end{bmatrix}$$

$$D=\begin{bmatrix}\frac{275}{20736}&-\frac{5717}{483840}&\frac{10621}{1088640}&-\frac{7703}{1451520}&\frac{403}{241920}&-\frac{199}{870912}\\\frac{97}{1890}&-\frac{2}{81}&\frac{197}{8505}&-\frac{97}{7560}&\frac{23}{5670}&-\frac{19}{34020}\\\frac{165}{1792}&-\frac{267}{17920}&\frac{5}{128}&-\frac{363}{17920}&\frac{57}{8960}&-\frac{47}{53760}\\\frac{376}{2835}&-\frac{2}{945}&\frac{656}{8505}&-\frac{2}{81}&\frac{8}{945}&-\frac{2}{1701}\\\frac{8375}{48384}&\frac{3125}{290304}&\frac{25625}{217728}&-\frac{625}{96768}&\frac{275}{20736}&-\frac{1375}{870912}\\3/14&\frac{3}{140}&\frac{17}{105}&\frac{3}{280}&\frac{3}{70}&0\end{bmatrix}.$$

',

The stability polynomial of the hybrid block methods is obtain as

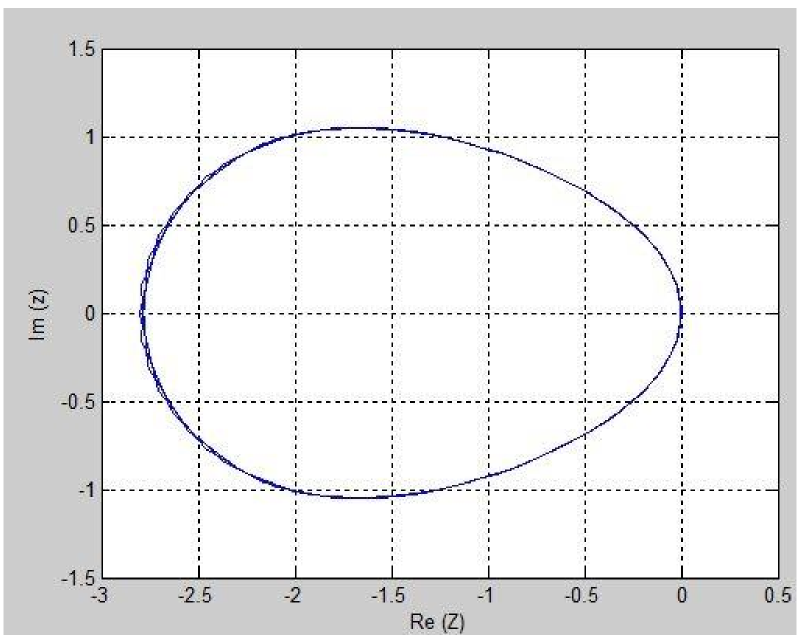

Fig. 1. Showing the region of absolute stability for (8)

$$fz=\frac{1}{426649337856}\eta^6z^{12}+\frac{167}{145118822400}\eta^6z^{11}+\frac{131}{338610585600}\eta^6z^{10}+\frac{3167}{6046617600}\eta^6z^9+\frac{89}{10581580800}\eta^6z^8+\frac{7961}{94058496}\eta^6z^7-\frac{67}{94058496}\eta^6z^6+\frac{1793}{311040}\eta^6z^5+\frac{41}{311040}\eta^6z^4+\frac{65}{432}\eta^6z^3-\frac{7}{432}\eta^6z^2+\eta^6z+\eta^6-\frac{782521}{71108222976000}\eta^5z^{10}-\frac{3095951}{740710656000}\eta^5z^8-\frac{631849}{1175731200}\eta^5z^6-\frac{58511}{2177280}\eta^5z^4-\frac{6577}{15120}\eta^5z^2-\eta^5$$

It's first derivatives is obtained as

$$fzp = \frac{1}{35554111488}\eta^6 z^{11} + \frac{1837}{145118822400}\eta^6 z^{10} + \frac{131}{33861058560}\eta^6 z^9 + \frac{3167}{671846400}\eta^6 z^8 + \frac{89}{1322697600}\eta^6 z^7 + \frac{7961}{13436928}\eta^6 z^6 - \frac{67}{15676416}\eta^6 z^5 + \frac{1793}{62208}\eta^6 z^4 + \frac{41}{77760}\eta^6 z^3 + \frac{65}{144}\eta^6 z^2 - \frac{7}{216}\eta^6 z + \eta^6 - \frac{782521}{7110822297600}\eta^5 z^9 - \frac{3095951}{92588832000}\eta^5 z^7 - \frac{631849}{195955200}\eta^5 z^5 - \frac{58511}{544320}\eta^5 z^3 - \frac{6577}{7560}\eta^5 z$$

$$N(z) = \frac{fz}{fzp}$$

plotting $N(z)$ via matlab environment displays the region of absolute stability in Fig. 1.

4 Implementation of the Method

In this section, we implement the proposed block method in solving initial value problems in ordinary differential equations. The proposed method is tested on some problems to determine the performance of the new proposed schemes and compared the results with other authors in the literature. The following problem below are tested.

Problem 1.

Consider a non linear differential equation (Source: [14])

$$y''(x) - x(y')^2 = 0, y(0) = 1, y'(0) = 0.5, h = 0.1$$

Exact Solution

$$y(x) = 1 + \frac{1}{2} ln(\frac{2+x}{2-x})$$

Table 1. Numerical and error results for problem 1

x	**Exact**	**Numerical Result**	**Error Result k=1, p=7**	**[14], k=3, p=8**
0.1	1.05004172927849126820	1.05004172927849085624	$4.1196000E-16$	$1.957046E-13$
0.2	1.10033534773107558060	1.10033534773107466662	$9.1398000E-16$	$6.039897E-13$
0.3	1.15114043593646680530	1.15114043593646517581	$1.6294900E-15$	$1.261598E-12$
0.4	1.20273255405408219100	1.20273255405407942497	$2.7660300E-15$	$3.715303E-12$
0.5	1.25541281188299534160	1.25541281188299062840	$4.7132000E-15$	$7.918892E-12$
0.6	1.30951960420311171550	1.309519604203103463134	$8.252370E-15$	$1.416178E-11$
0.7	1.36544375427139616910	1.36544375427138112957	$1.503953E-14$	$3.616015E-11$
0.8	1.42364893019360180680	1.42364893019357301828	$2.878852E-14$	$7.472525E-11$
0.9	1.48470027859405174160	1.48470027859399336945	$5.837215E-14$	$1.335141E-10$
1.0	1.54930614433405484570	1.54930614433392823951	$1.2660619E-13$	$4.316861E-10$

Problem 2.

Consider a system of equation of the form (Source: [7])

$y''(x) = -e^{-x}y_2, y_1(0) = 1, y'(0) = 0, h = 0.1, y''(x) = 2e^x y'_1$, $y_2(0) = 1$, $y_2'(0) = 1$, Exact Solution

$$y_1(x) = \cos(x), y_2(x) = e^x \cos(x)$$

Table 2. Numerical and Error Results for Problem 2 of y_1

x	Exact	Numerical Result of y_1	Error Result k=1,p=7 of y_1	[6], k=1,p=5 of y_1
0.2	0.980066577841241630	0.980066577841242099	$4.6900E-16$	$3.348466E-09$
0.4	0.921060994002884990	0.921060994002887760	$2.7700E-15$	$3.276545E-08$
0.6	0.825335614909678110	0.825335614909685644	$7.5340E-15$	$1.332214E-07$
0.8	0.69670670934716505	0.696706709347180047	$1.4997E-14$	$3.546280E-07$
1.0	0.540302305868139210	0.540302305868163875	$2.4665E-14$	$7.355177E-07$

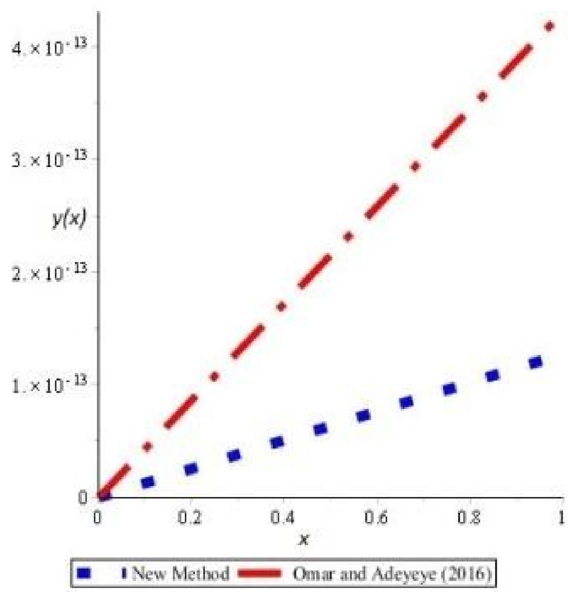

Fig. 2. Showing the Maximum errors of problem 1

Table 3. Numerical and Error Results for Problem 2 of y_2

x	Exact	Numerical Result of y_2	Error Result k=1,p=7 of y_2	[7], k=1,p=5 of y_2
0.2	1.197056021355891400	1.19705602135584996	$4.144000E-14$	$6.304139E-07$
0.4	1.374061538887522100	1.37406153888744350	$7.860000E-14$	$2.521669E-06$
0.6	1.503859540558786200	1.50385954055868131	$1.048900E-13$	$5.429593E-06$
0.8	1.550549296807422400	1.55054929680731085	$1.115500E-13$	$8.852781E-08$
1.0	1.468693939915884900	1.46869393991579831	$8.659000E-14$	$1.194695E-08$

Problem 3.

Consider an oscillatory non-linear system of initial value problems (Source: [14])

$$y''(x) = -4x^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, y_1(0) = 1, y_1'(0) = 0, h = 0.125$$
$$y''(x) = -4x^2 y_2 + \frac{2y_1}{\sqrt{y_1^2 + y_2^2}}, y_2(0) = 0, y_2'(0) = 0$$

Exact Solution

$y_1(x) = \cos(x^2), y_2(x) = \sin(x^2)$

Table 4. Numerical and Error Results for Problem 3

−	Maximum error of y_1	Minimum error of y_1	Maximum error of y_2	Minimum error of y_2
H	$4.461721215E-12$	$1.145493E-15$	$1.17577969E-12$	$6.28692460E-14$
x	0.875	0.125	0.875	0.125

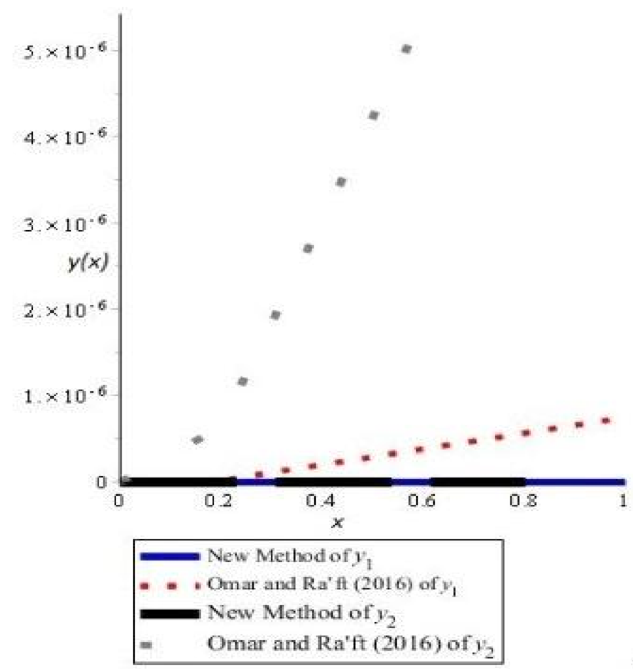

Fig. 3. Showing the Maximum and Minimum errors of problem 2

5 Discussion and Conclusion

The linear multi-step method (8) proposed in this paper, was analyzed and found to be of order p = 7, consistent, zero-stable, convergent with good region of absolute stability. From Fig. 1, it is observed that the region of absolute stability has negative real number, which implies that the roots $\lambda < 0$. This therefore make the numerical results in Tables 1 and 3 to decay faster for $h > 0$ in linear and non linear IVPs when compared to that of [14], even thought their method is of order $p = 8$ with step number $k = 3$ against ours. The region of absolute stability of their method contained both negative and positive real part, this decay for sufficiently small value of h and converge faster for only linear IVP. While, in Table 2, our method still compete with that of [7], even though both methods are of the same step number but different order and their region of absolute stability contained all positive real part. Figs. 2-3 shows the maximum errors of the proposed method.

In conclusion, we have derived one-step hybrid block method for solution of second order initial value problems without reducing to system of first order ordinary differential equation. The numerical and graphical results shows that the proposed method perform better than the compared methods

Acknowledgement

We wish to acknowledge Professor R. B. Adeniyi of Mathematics Department University of Ilorin, Nigeria for introducing us to the concept of linear multi-step methods.

Competing Interests

Authors have declared that no competing interests exist.

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