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Some Stationary Solutions of Schrödinger Map Equation

Yanting Zhou[1], Hui Yang[1] and Ganshan Yang[1,2*]

[1]*Institute of Mathematics, Yunnan Normal University, Kunming, 650500, China.*
[2]*Department of Mathematics, Yunnan Minzu University, Kunming, 650504, China.*

Authors' contributions

This work was carried out in collaboration between all authors. Author YZ designed the study some stationary solutions of Schrödinger Map Equation, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors HY and GY managed the analyses of the study and the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, we construct a type of plane wave solution of Landau-Lifshitz equation with the model $|u| = 1$. In addition, we discover the law which when the spin vector u is moving along one direction, the spin vector u approaches the south pole $(0, 0, -1)$ from the north pole $(0, 0, 1)$ with the model $|x|$ from 0 tend to ∞. Landau-Lifshitz equations describe an evolution of spin field in continuous ferromagnetic. Therefore, it is very significant to study the problems about magnetization movement. Many people studied a lot of problems and constructed many solutions about the Landau-Lifshitz equation, but no one to study the linear plane wave solution. So, in this paper, we construct some stationary solutions of Schrodinger Map equation which contains a style of plane wave solutions.

**Corresponding author: E-mail: yangganshan@aliyun.com;*
E-mail: 1486300915@qq.com;

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1 Introduction

Landau and Lifshitz [1] proposed the equation

$$\begin{cases} \frac{\partial u}{\partial t} = \lambda_1 u \times H^e - \lambda_2 u \times (u \times H^e), \quad \Omega \times (0,T), \\ H^e := -H(u) + \Sigma_{i,j=1}^{3} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + H, \quad \Omega \times (0,T), \end{cases} \tag{1.1}$$

where λ_1 and λ_2 are constants with $\lambda_2 > 0$, the three-dimensional vector $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ means the evolution of the magnetization vector with $t \in (0,T)$ and $T > 0$, the n-dimensional vector $x \in \Omega \subset R^n$ remarks the physical particle of magnet, H^e is general effective field. This equation describes an evolution of spin fields in continuous ferromagnetic and bears a fundamental role in the understanding of non-equilibrium, just as the Navier-Stokes equation does in that of fluid dynamics.

It is also very difficult to find the exact solution of the Landau-Lifshitz equation with external field, especially, the case when the evolution of the magnetization vector u with $|u| = 1$. In recent years, more and more physicists and mathematicians studied and did a lot of work on the Landau-Lifshitz equations. Ding and Guo [2] given the existence, partial regularity and uniqueness of weak solution to the initial boundary value problem for the unsaturated Landau-Lifshitz systems. Ding and Wang [3] proved that in dimensions three or four, for suitably chosen initial data, the short time smooth solution to the Landau-Lifshitz-Gilbert equation blows up at finite time. Zhong, [4] constructed the exact solution of two or three-dimensional space time Landau-Lifshitz equation raised in the ferromagnetic materials, under suitable transformations, some exact solutions are obtained in the radially symmetric coordinates and the type of solution covered the finite time blow-up solution, vortex solution and periodic solution. Zhong, etc. [5] constructed two exact blowup solutions of the $(2 + 1)$-dimensional space-time inhomogeneous isotropic Landau-Lifshitz equation under suitable transformations.

Considering Landau-Lifshitz equation (1.1) without dynamical damp ($\lambda_2 = 0$), and without external magnetic field, suppose $\lambda_1 = 1$, we have

$$\begin{cases} u_t = u \times \Delta u, \\ u \in \mathbb{S}^2, \end{cases} \tag{1.2}$$

which means the intensity of magnetization u moving around the effective field Δu. It is famous Heisenberg spin system [6] and so-called Schrödinger map (SM) equation in the geometry [7].

As regard SM equation of one-dimensional, Zhou, etc. [8] and Sulem, etc. [9] studied the Cauchy problem respectively and proved the global existence of the weak solution and global existent unique of small data smooth solution. Chang, etc. [10] and Ding, etc. [11] proved the global existent unique of one dimension smooth solution. For the Cauchy problem of two dimensions case, Chang, etc. [10] had got the existent unique of small data global smooth solution. In order to avoid the limit of small data, Gustafson, etc. [12] analyzed the existence of solitary wave solution, they also analysed the anisotropic SM equation

$$u_t = u \times (\Delta u + \lambda u_3(0,0,1)), \quad \lambda > 0, \tag{1.3}$$

and obtained unique global smooth solution of Cauchy problem with initial value is equivalent or localized. About multidimensional case, Yang, etc. [13, 14, 15] constructed some explicit solutions.

There is an interesting result, Guo, etc. [16] constructed non-trivial global smooth solution of SM equation and the energy of initial value is arbitrary large, which is no limit of small energy.

However, many people studied a lot of problems and construct many solutions about the LL equation, but no one to study the linear plane wave solution. So, in this paper, we construct some stationary solutions of Schrödinger map equation which contain a style linear plane wave solutions. and obtain some dynamical solutions of the equation

$$\begin{cases} v_t = v \times (\Delta v + (0,0,h)), \\ v \in \mathbb{S}^2. \end{cases} \tag{1.4}$$

where constant vector $(0,0,h)$ is external magnetic field. In addition, we discover the law which when the intensity of magnetization u is moving along one direction, the intensity of magnetization u approaches the south pole $(0,0,-1)$ from the north pole $(0,0,1)$ with the model $|x|$ from 0 tend to ∞.

2 Stationary Solutions of SM Equation

In this section, we consider the stationary solutions of the equation (1.2):

$$\begin{cases} u \times \triangle u = 0, \\ u \in \mathbb{S}^2. \end{cases} \tag{2.1}$$

which has the form

$$u = (u_1, u_2, u_3) = \frac{1}{1+f_1^2+f_2^2}\begin{bmatrix} 2f_1^2 \\ 2f_2^2 \\ 1-f_1^2-f_2^2 \end{bmatrix}, \tag{2.2}$$

where $f_1 = f_1(x), f_2 = f_2(x)$ and $x = (x_1, x_2, \cdots, x_n)$. We compute the term Δu firstly,

$$\Delta u = u_{f_1f_1}|Df_1|^2 + u_{f_2f_2}|Df_2|^2 + 2u_{f_1f_2}Df_1Df_2 + u_{f_1}\Delta f_1 + u_{f_2}\Delta f_2. \tag{2.3}$$

Substituting (2.3) into the equation (2.1), we have

$$u\times\Delta u = u\times u_{f_1f_1}|Df_1|^2 + u\times u_{f_2f_2}|Df_2|^2 + u\times 2u_{f_1f_2}Df_1Df_2 + u\times u_{f_1}\Delta f_1 + u\times u_{f_2}\Delta f_2. \tag{2.4}$$

Next we calculate the terms of the equation (2.4),

$$\begin{cases} u\times u_{f_1f_1} = \frac{-8f_1}{1+f_1^2+f_2^2}u\times\begin{pmatrix} 1 \\ 0 \\ -f_1 \end{pmatrix} + \frac{1}{1+f_1^2+f_2^2}u\times\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \\ u\times u_{f_2f_2} = \frac{-8f_2}{1+f_1^2+f_2^2}u\times\begin{pmatrix} 1 \\ 0 \\ -f_2 \end{pmatrix} + \frac{1}{1+f_1^2+f_2^2}u\times\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \\ u\times u_{f_1f_2} = \frac{-4f_1}{(1+f_1^2+f_2^2)^2}u\times\begin{pmatrix} 1 \\ 0 \\ -f_2 \end{pmatrix} + \frac{-4f_2}{(1+f_1^2+f_2^2)^2}u\times\begin{pmatrix} 1 \\ 0 \\ -f_1 \end{pmatrix}. \end{cases} \tag{2.5}$$

Substituting (2.5) into the equation (2.4), we have

$$
\begin{aligned}
0 = {} & \frac{-8f_1|Df_1|^2}{(1+f_1^2+f_2^2)^3}\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \frac{|Df_1|^2}{(1+f_1^2+f_2^2)^2}\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \\
& \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \frac{-8f_2|Df_2|^2}{(1+f_1^2+f_2^2)^3}\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -f_2 \end{pmatrix} + \frac{|Df_2|^2}{(1+f_1^2+f_2^2)^2} \\
& \begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \frac{-8f_1Df_1Df_2}{(1+f_1^2+f_2^2)^3}\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -f_2 \end{pmatrix} + \\
& \frac{-8f_1Df_1Df_2}{(1+f_1^2+f_2^2)^3}\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -f_1 \end{pmatrix} + \frac{\Delta f_1}{(1+f_1^2+f_2^2)^2}\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \\
& \begin{pmatrix} 2 \\ 0 \\ -2f_1 \end{pmatrix} + \frac{\Delta f_2}{(1+f_1^2+f_2^2)^2}\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ -2f_2 \end{pmatrix}.
\end{aligned} \tag{2.6}
$$

Multiplying the formula $(1+f_1^2+f_2^2)^3$ in both sides of above formula, then (2.6) is equivalent to

$$
\begin{aligned}
0 = {} & -8f_1|Df_1|^2\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -f_1 \end{pmatrix} + |Df_1|^2(1+f_1^2+f_2^2)\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \\
& \times \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} - 8f_2|Df_2|^2\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -f_2 \end{pmatrix} + |Df_2|^2(1+f_1^2+f_2^2) \\
& \begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} - 8f_1 \cdot Df_1 \cdot Df_2\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -f_2 \end{pmatrix} \\
& - 8f_2 \cdot Df_1 \cdot Df_2\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -f_1 \end{pmatrix} + \Delta f_1(1+f_1^2+f_2^2)\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \\
& \times \begin{pmatrix} 2 \\ 0 \\ -2f_1 \end{pmatrix} + \Delta f_2(1+f_1^2+f_2^2)\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1^2-f_2^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ -2f_2 \end{pmatrix}.
\end{aligned} \tag{2.7}
$$

Following the proposition of the cross product, there exist g such that the equation (2.7) can rewrite as following:

$$
\begin{aligned}
& \begin{pmatrix} 1 \\ 0 \\ -f_1 \end{pmatrix}(\Delta f_1(1+f_1^2+f_2^2) - 4f_1|Df_1|^2) - (4f_2|Df_2|^2 + 4f_1 \cdot Df_1 \cdot Df_2)\begin{pmatrix} 0 \\ 1 \\ -f_2 \end{pmatrix} \\
& + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}(1+f_1^2+f_2^2)(|Df_1|^2 + |Df_2|^2) - 4f_1 \cdot Df_1 \cdot Df_2\begin{pmatrix} 1 \\ 0 \\ -f_1 \end{pmatrix} + \\
& \Delta f_2(1+f_1^2+f_2^2)\begin{pmatrix} 0 \\ 1 \\ -f_2 \end{pmatrix} = -g\begin{pmatrix} 2f_1 \\ 2f_2 \\ 1-f_1-f_2 \end{pmatrix}.
\end{aligned} \tag{2.8}
$$

That is

$$\begin{cases} -4f_1|Df_1|^2 + \Delta f_1(1 + {f_1}^2 + {f_2}^2) - 4f_2 \cdot Df_1 \cdot Df_2 = -2gf_1 \\ -4f_2|Df_2|^2 + \Delta f_2(1 + {f_1}^2 + {f_2}^2) - 4f_1 \cdot Df_1 \cdot Df_2 = -2gf_2 \\ (-f_1\Delta f_1 - |Df_1|^2 - |Df_2|^2 - f_2\Delta f_2)(1 + {f_1}^2 + {f_2}^2) + 4{f_1}^2|Df_1|^2 \\ +4{f_2}^2|Df_2|^2 + 8f_1f_2 \cdot Df_1 \cdot Df_2 = -g(1 - {f_1}^2 - {f_2}^2). \end{cases} \tag{2.9}$$

After simplifying, we can get the following formula:

$$|Df_1|^2 + |Df_2|^2 = g. \tag{2.10}$$

Thus, the equation (2.9) is equivalent to the following equations:

$$\begin{cases} |Df_1|^2 + |Df_2|^2 = g \\ 4f_1|Df_1|^2 - \Delta f_1(1 + {f_1}^2 + {f_2}^2) + 4f_2 \cdot Df_1 \cdot Df_2 = 2gf_1 \\ 4f_2|Df_2|^2 - \Delta f_2(1 + {f_1}^2 + {f_2}^2) + 4f_1 \cdot Df_1 \cdot Df_2 = 2gf_2. \end{cases} \tag{2.11}$$

We can simplify the above equations, and we get the following equations:

$$\begin{cases} 2f_1|Df_1|^2 - \Delta f_1(1 + {f_1}^2 + {f_2}^2) + 4f_2 \cdot Df_1 \cdot Df_2 - 2f_1|Df_2|^2 = 0 \\ 2f_2|Df_2|^2 - \Delta f_2(1 + {f_1}^2 + {f_2}^2) + 4f_1 \cdot Df_1 \cdot Df_2 - 2f_2|Df_1|^2 = 0. \end{cases} \tag{2.12}$$

Next we construct the Plane wave solutions of equation (2.12). Assuming that $f_1 = \Sigma_{i=1}^n a_i x_i$, and $f_2 = \Sigma_{i=1}^n b_i x_i$, then the equations (2.12) can be rewrote as following:

$$\begin{cases} 2\Sigma_{i=1}^n a_i x_i \cdot \Sigma_{i=1}^n {a_i}^2 + 4\Sigma_{i=1}^n b_i x_i \cdot \Sigma_{i=1}^n a_i b_i - 2\Sigma_{i=1}^n a_i x_i \cdot \Sigma_{i=1}^n {b_i}^2 = 0 \\ 2\Sigma_{i=1}^n b_i x_i \cdot \Sigma_{i=1}^n {b_i}^2 + 4\Sigma_{i=1}^n b_i x_i \cdot \Sigma_{i=1}^n a_i b_i - 2\Sigma_{i=1}^n b_i x_i \cdot \Sigma_{i=1}^n {a_i}^2 = 0 \end{cases}. \tag{2.13}$$

Let the vector $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n)$, then we rewrote the equation (2.13) as following:

$$\begin{cases} a_i(A^2 - B^2) + 2b_i AB = 0 \\ b_i(B^2 - A^2) + 2a_i AB = 0, \quad i = 1, \ldots, n. \end{cases} \tag{2.14}$$

According to the Cramer's rule, the equations (2.14) has unique solution, which is

$$\begin{cases} AB = 0, \\ A^2 = B^2. \end{cases}$$

i.e.

$$\begin{cases} \Sigma_{i=1}^n a_i b_i = 0, \\ \Sigma_{i=1}^n {b_i}^2 = {a_i}^2. \end{cases} \tag{2.15}$$

2.1 Considering the two dimensional case $(n = 2)$

In this section, we consider the two-dimensional case.

(i) If $A = (a, b), a, b \in \mathbb{R}$, then $B = (-b, a)$ or $B = (b, -a)$. We can obtain the following equations:

$$\begin{cases} f_1 = ax_1 + bx_2, \\ f_2 = -bx_1 + ax_2, \end{cases}$$

or

$$\begin{cases} f_1 = ax_1 + bx_2, \\ f_2 = bx_1 - ax_2. \end{cases}$$

Thus, the solution of the equation (2.1) can be wrote as following:

$$g_1 = \frac{1}{1 + (a^2 + b^2)({x_1}^2 + {x_2}^2)}(2ax_1 + 2bx_2, 2ax_2 - 2bx_1, 1 - (a^2 + b^2)({x_1}^2 + {x_2}^2)), \tag{2.16}$$

or

$$g_2 = \frac{1}{1 + (a^2 + b^2)({x_1}^2 + {x_2}^2)}(2ax_1 + 2bx_2, 2bx_1 - 2ax_2, 1 - (a^2 + b^2)({x_1}^2 + {x_2}^2)). \tag{2.17}$$

(ii) If $A = (a, ia), a \in \mathbb{R}$, then $B = (-ia, a)$ or $B = (ia, -a)$, we have

$$\begin{cases} f_1 = ax_1 + iax_2, \\ f_2 = -iax_1 + ax_2, \end{cases}$$

or

$$\begin{cases} f_1 = ax_1 + iax_2, \\ f_2 = iax_1 - ax_2. \end{cases}$$

So, the solutions of the equation (2.1) can be wrote as following:

$$g_3 = (2ax_1 + 2iax_2, -2iax_1 + 2ax_2, 1), \tag{2.18}$$

or

$$g_4 = (2ax_1 + 2iax_2, 2iax_1 - 2ax_2, 1). \tag{2.19}$$

(iii) If $A = (ia, a), a \in \mathbb{R}$, then $B = (a, -ia)$ or $B = (-a, ia)$. We have:

$$\begin{cases} f_1 = iax_1 + ax_2, \\ f_2 = ax_1 - iax_2, \end{cases}$$

or

$$\begin{cases} f_1 = iax_1 + ax_2, \\ f_2 = -ax_1 + iax_2. \end{cases}$$

So, the solutions of the equation (2.1) can be wrote as following:

$$g_5 = (2iax_1 + 2ax_2, 2ax_1 - 2iax_2, 1), \tag{2.20}$$

or

$$g_6 = (2iax_1 + 2ax_2, -2ax_1 + 2iax_2, 1). \tag{2.21}$$

2.2 Considering the three dimensional case $(n = 3)$

In this section, we consider the three-dimensional case. Assume that the vector $A = (a_1, b_1, c_1)$, $a_1, b_1, c_1 \in \mathbb{R}$, substituting it into (2.15), we have

$$B = (\frac{{b_1}^2c_1c_2 - b_1M^{\frac{1}{2}} - ({a_1}^2 + {b_1}^2)c_1c_2}{({a_1}^2 + {b_1}^2)a_1}, \frac{M^{\frac{1}{2}} - b_1c_1c_2}{{a_1}^2 + {b_1}^2}, c_2),$$

where c_2 is arbitrary constant and

$$M = {a_1}^6 + 2{a_1}^4{b_1}^2 + {a_1}^4{c_1}^2 + {a_1}^2{b_1}^4 - {a_1}^4{c_2}^2 + {a_1}^2{b_1}^2{c_1}^2 - {a_1}^2{b_1}^2{c_2}^2 - {a_1}^2{c_1}^2{c_2}^2,$$

Therefore,

$$\begin{cases} f_1 = a_1x_1 + b_1x_2 + c_1x_3, \\ f_2 = \frac{{b_1}^2c_1c_2 - b_1M^{\frac{1}{2}} - ({a_1}^2 + {b_1}^2)c_1c_2}{({a_1}^2 + {b_1}^2)a_1}x_1 + \frac{-b_1c_1c_2 + M^{\frac{1}{2}}}{{a_1}^2 + {b_1}^2}x_2 + c_2x_3. \end{cases} \tag{2.22}$$

We can substitute the equality (2.22) into the equation (2.2) and obtain the solution g_7 of the equation (2.1), that is

$$\begin{aligned} g_7 = &\Bigg(1 + (a_1x_1 + b_1x_2 + c_1x_3)^2 + (\frac{{b_1}^2c_1c_2x_1}{({a_1}^2 + {b_1}^2)a_1} - \frac{c_1c_2x_1}{a_1} - \frac{b_1c_1c_2x_2}{{a_1}^2 + {b_1}^2} + c_2x_3 - \\ &\frac{b_1M^{\frac{1}{2}}x_1}{({a_1}^2 + {b_1}^2)a_1} + \frac{M^{\frac{1}{2}}x_2}{{a_1}^2 + {b_1}^2})^2\Bigg)^{-1}\Bigg(2(a_1x_1 + b_1x_2 + c_1x_3), 2(\frac{M^{\frac{1}{2}}}{(a_1^2 + b_1^2)a_1}x_1 + c_2x_3 \\ &+ \frac{c_1c_2}{a_1}x_1 + \frac{-b_1c_1c_2 + M^{\frac{1}{2}}}{a_1^2 + b_1^2}x_2 - \frac{b_1c_1c_2}{(a_1^2 + b_1^2)a_1}x_1), 1 - (a_1x_1 + b_1x_2 + c_1x_3)^2 - \\ &(\frac{M^{\frac{1}{2}} - b_1c_1c_2 + c_1c_2(a_1^2 + b_1^2)}{(a_1^2 + b_1^2)a_1}x_1 + \frac{-b_1c_1c_2 + M^{\frac{1}{2}}}{a_1^2 + b_1^2}x_2 + c_2x_3)^2\Bigg). \end{aligned} \tag{2.23}$$

We assume

$$M = {a_1}^6 + 2{a_1}^4{b_1}^2 + {a_1}^4{c_1}^2 + {a_1}^2{b_1}^4 - {a_1}^4{c_2}^2 + {a_1}^2{b_1}^2{c_1}^2 - {a_1}^2{b_1}^2{c_2}^2 - {a_1}^2{c_1}^2{c_2}^2.$$

After computing by the Maple Soft, we can prove the conclusion that all solutions above are satisfied the equation (2.1). Next we will consider the dynamical solutions of SM equation with external magnetic field $(0, 0, h)$.

3 Dynamical Solutions of SM Equation with External Magnetic Field

Now, we consider the dynamical solutions of the SM equation with external magnetic field $(0, 0, h)$. In order to construct the solutions of equation (1.4), we firstly exhibit the following Lemma.

Lemma 3.1. *If u is the stationary solution of the equation (2.1), then $v = uQ$ is the solution of the equation (1.4), where Q is the following first class Orthogonal matrix:*

$$\begin{pmatrix} \cos ht & -\sin ht & 0 \\ \sin ht & \cos ht & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Substituting $v = uQ$ into the equation (1.4), we have

$$\begin{aligned} v \times (\Delta v + (0,0,h)) &= v \times \Delta v + v \times (0,0,h) \\ &= u \cdot Q \times \Delta v + h \cdot u \cdot Q \times (0,0,1) \\ &= (u \times \Delta u)Q + h \cdot u \cdot Q \times (0,0,1) \\ &= h \cdot u \cdot Q \times (0,0,1) \\ &= h(u_1,u_2,u_3)\begin{pmatrix} \cos ht & -\sin ht & 0 \\ \sin ht & \cos ht & 0 \\ 0 & 0 & 1 \end{pmatrix} \times (0,0,1) \\ &= (-hu_1 \sin ht + hu_2 \cos ht, -hu_1 \cos ht - hu_2 \sin ht, 0), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} v_t &= \left((u_1,u_2,u_3)\begin{pmatrix} \cos ht & -\sin ht & 0 \\ \sin ht & \cos ht & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)_t \\ &= (u_1,u_2,u_3)\begin{pmatrix} \cos ht & -\sin ht & 0 \\ \sin ht & \cos ht & 0 \\ 0 & 0 & 1 \end{pmatrix}_t \\ &= (u_1,u_2,u_3)\begin{pmatrix} -h\sin ht & -h\cos ht & 0 \\ h\cos ht & -h\sin ht & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-hu_1 \sin ht + hu_2 \cos ht, -hu_1 \cos ht - hu_2 \sin ht, 0). \end{aligned} \tag{3.2}$$

Thus $v = uQ$ is the solution of the equation (1.4). □

According to the above lemma, we can get the following solutions of the equation (1.4):

$$\begin{aligned} v_1 =& \frac{1}{1+(a^2+b^2)(x_1^2+x_2^2)}\big((2ax_1+2bx_2)\cos ht + (2ax_2-2bx_1)\sin ht, \\ & -(2ax_1+2bx_2)\sin ht + (2ax_2-2bx_1)\cos ht, 1-(a^2+b^2)({x_1}^2+{x_2}^2)\big), \\ v_2 =& \frac{1}{1+(a^2+b^2)(x_1^2+x_2^2)}\big((2ax_1+2bx_2)\cos ht + (2bx_1-2ax_2)\sin ht, \\ & -(2ax_1+2bx_2)\sin ht + (2bx_1-2ax_2)\cos ht, 1-(a^2+b^2)({x_1}^2+{x_2}^2)\big), \\ v_3 =& \big((2ax_1+2iax_2)\cos ht + (2ax_2-2iax_1)\sin ht, -(2ax_1+2iax_2)\sin ht \\ & + (2ax_2-2iax_1)\cos ht, 1\big), \end{aligned}$$

$$v_4 = \big((2ax_1 + 2iax_2)\cos ht + (2iax_1 - 2ax_2)\sin ht, -(2ax_1 + 2iax_2)\sin ht \\ + (2iax_1 - 2ax_2)\cos ht, 1\big),$$

$$v_5 = \big((2iax_1 + 2ax_2)\cos ht + (2ax_1 - 2iax_2)\sin ht, -(2iax_1 + 2ax_2)\sin ht \\ + (2ax_1 - 2iax_2)\sin ht, 1\big),$$

$$v_6 = \big((2iax_1 + 2ax_2)\cos ht + (-2ax_1 + 2iax_2)\sin ht, -(2iax_1 + 2ax_2)\sin ht \\ + (-2ax_1 + 2iax_2)\cos ht, 1\big),$$

and $u_7 \cdot Q$ is also the solution of the equation (1.4), that is

$$v_7 = \Bigg(1 + (a_1x_1 + b_1x_2 + c_1x_3)^2 + \Big(\frac{{b_1}^2c_1c_2x_1}{({a_1}^2+{b_1}^2)a_1} - \frac{c_1c_2x_1}{a_1} - \frac{b_1c_1c_2x_2}{{a_1}^2+{b_1}^2} + c_2x_3 \\ - \frac{b_1M^{\frac{1}{2}}x_1}{({a_1}^2+{b_1}^2)a_1} + \frac{M^{\frac{1}{2}}x_2}{{a_1}^2+{b_1}^2}\Big)^2\Bigg)^{-1}\Bigg(2\sin ht\Big(\frac{M^{\frac{1}{2}}}{(a_1^2+b_1^2)a_1}x_1 + c_2x_3 + \frac{c_1c_2}{a_1}x_1 \\ \frac{-b_1c_1c_2 + M^{\frac{1}{2}}}{a_1^2+b_1^2}x_2 - \frac{b_1c_1c_2}{(a_1^2+b_1^2)a_1}x_1\Big) + 2\cos ht(a_1x_1 + b_1x_2 + c_1x_3), \\ -2\sin ht(a_1x_1 + b_1x_2 + c_1x_3) + 2\cos ht\Big(\frac{M^{\frac{1}{2}}}{(a_1^2+b_1^2)a_1}x_1 + c_2x_3 + \frac{c_1c_2}{a_1}x_1 \\ + \frac{-b_1c_1c_2 + M^{\frac{1}{2}}}{a_1^2+b_1^2}x_2 - \frac{b_1c_1c_2}{(a_1^2+b_1^2)a_1}x_1\Big), 1 - (a_1x_1 + b_1x_2 + c_1x_3)^2 - \\ \Big(\frac{M^{\frac{1}{2}} - b_1c_1c_2 + c_1c_2(a_1^2+b_1^2)}{(a_1^2+b_1^2)a_1}x_1 + \frac{-b_1c_1c_2 + M^{\frac{1}{2}}}{a_1^2+b_1^2}x_2 + c_2x_3\Big)^2\Bigg).$$

We assume

$$M = {a_1}^6 + 2{a_1}^4{b_1}^2 + {a_1}^4{c_1}^2 + {a_1}^2{b_1}^4 - {a_1}^4{c_2}^2 + {a_1}^2{b_1}^2{c_1}^2 - {a_1}^2{b_1}^2{c_2}^2 - {a_1}^2{c_1}^2{c_2}^2.$$

4 The Figures of the Solutions of the Schrödinger Map Equation

Firstly, we construct the figures of the stationary solutions g_1 and help us find out the law of the solution of the Schrödinger map equation. We assume that the parameters of g_1, $a = \frac{\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}$. Therefore, the solution g_1 becomes as the following formula: $g_1 = \frac{1}{1+({x_1}^2+{x_2}^2)}(\sqrt{2}x_1 + \sqrt{2}x_2, -\sqrt{2}x_1 + \sqrt{2}x_2, 1 - ({x_1}^2 + {x_2}^2))$. Then we assume $x_2 = kx_1$ and observe the phenomenon that when $k = 1, |u| = 1$, $|x| \leq 0.001$, $|x| \leq 0.1$, $|x| \leq 1$, $|x| \leq 100$:

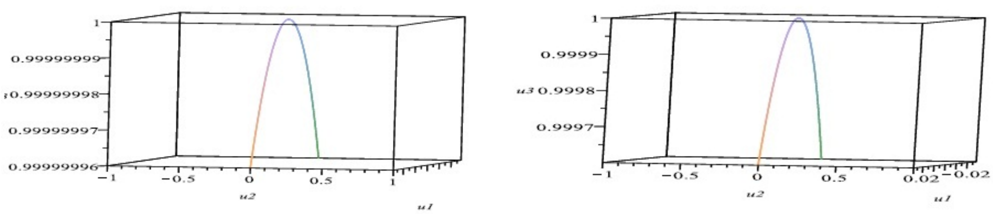

Fig. 1. *Let* $x_2 = kx_1$, $k = 1$, $|x_1| \leq 0.0001$ *(left figure), and* $|x_1| \leq 0.01$ *(right figure)*

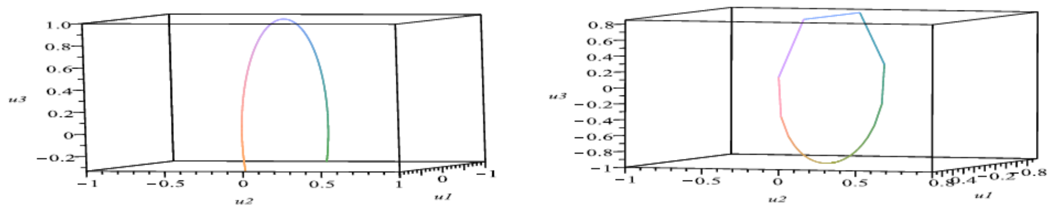

Fig. 2. *Let* $x_2 = kx_1,\ |x_1| \leq 1\,(left\,figure),\ and\,|x_1| \leq 100\,(right\,figure)$

From above figures we can discover the phenomenon that the intensity of magnetization g_1 approaches the north pole $(0, 0, 1)$ along the curve $x_2 = kx_1, k \in R$ when the model $|x|$ tend to 0. Another aspect, the intensity of magnetization g_1 approaches the south pole $(0, 0, -1)$ along the same curve $x_2 = kx_1, k \in R$ when the model $|x|$ tend to ∞.

Then we study again the case when $x_2 = k{x_1}^2$, assume $a = \frac{\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}, k = 1,\ |u| = 1,\ |x_1| \leq$ 0.0001, 1, 2, 4, 10, 20 and we describe those orbits as following:

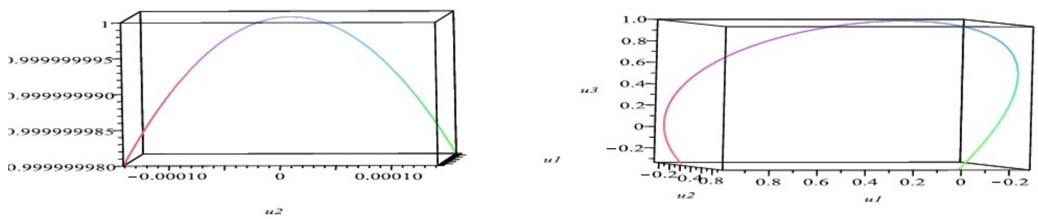

Fig. 3. *Let* $|x_1| \leq 0.0001\,(left\,figure),\ and\,|x_1| \leq 1\,(right\,figure)$

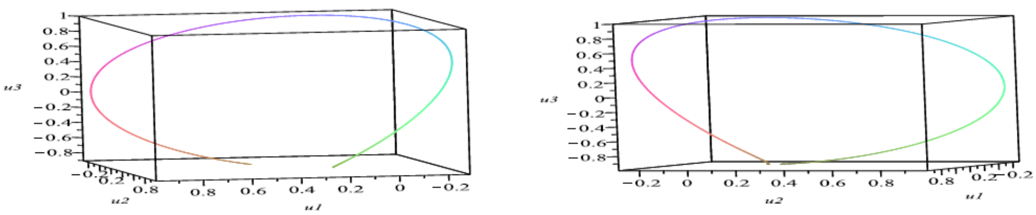

Fig. 4. *Let* $|x_1| \leq 2\,(left\,figure),\ and\,|x_1| \leq 4\,(right\,figure)$

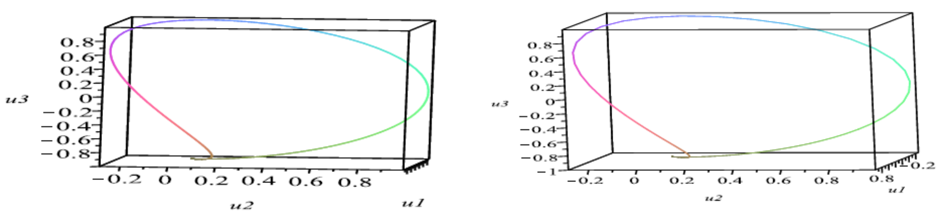

Fig. 5. *Let* $x_2 = k{x_1}^2,\ |x_1| \leq 10\,(left\,figure),\ and\,|x_1| \leq 20\,(right\,figure)$

From above figures we can find that the intensity of magnetization g_1 approaches the north pole $(0, 0, 1)$ along the curve $x_2 = kx_1{}^2, k \in R$ when the model $|x|$ tend to 0. Another aspect, the intensity of magnetization g_1 approaches the south pole $(0, 0, -1)$ when the model $|x|$ tend to ∞.

Next we study the different cases when $x_2 = kx_1$, $x_2 = kx_1{}^2$, and $x_2 = kx_1{}^3, k \in R$, and assume $a = \frac{\sqrt{2}}{2}$, $b = \frac{\sqrt{2}}{2}$, $k = 3$, $|x_1| \leq 2$, the solution : $u = \frac{1}{1+(x_1{}^2+x_2{}^2)}(\sqrt{2}x_1 + \sqrt{2}x_2, -\sqrt{2}x_1 + \sqrt{2}x_2, 1 - (x_1{}^2 + x_2{}^2))$, and we describe those orbits as following:

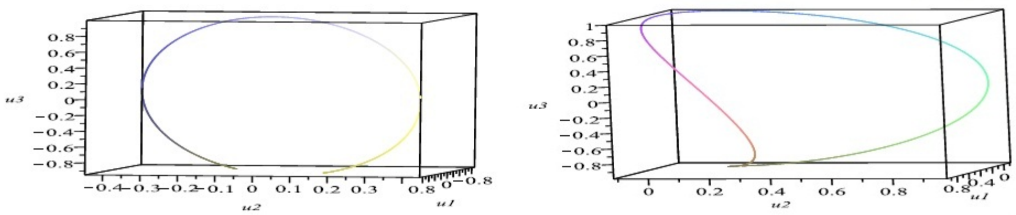

Fig. 6. $x_2 = kx_1,\, (left)\, and\, x_2 = kx_1{}^2,\, |x_1| \leq 2, (right)$

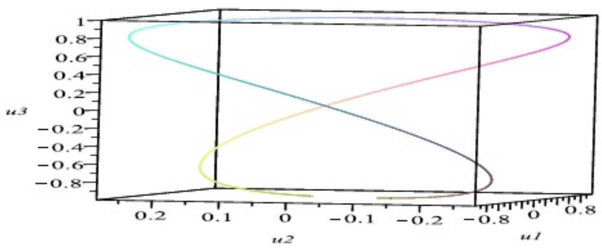

Fig. 7. $x_2 = kx_1{}^3,\, k = 3,\, |x_1| \leq 2$

Finally, we construct figures of the dynamical solutions v_1 and help us find out the logic of the solution of the Schrödinger map equation. We assume $a = \frac{\sqrt{2}}{2}$, $b = \frac{\sqrt{2}}{2}$, $h = 1$. Therefore, the solution v_1 becomes as the following formula:

$$\begin{aligned} v_1 =& \frac{1}{1+(x_1^2+x_2^2)}\big((\sqrt{2}x_1 + \sqrt{2}x_2)\cos t + (-\sqrt{2}x_1 + \sqrt{2}ax_2)\sin t, \\ & -(\sqrt{2}x_1 + \sqrt{2}x_2)\sin t + (-\sqrt{2}x_1 + \sqrt{2}x_2)\cos t, 1 - (x_1{}^2 + x_2{}^2)\big), \end{aligned}$$

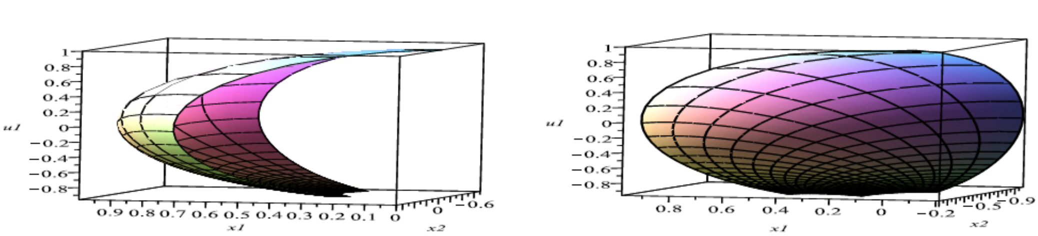

Fig. 8. $Let\, t = 0, x_1 \in (0, 5),\, x_2 \in (0, 5)\, (left)\, and\, t = 1,\, x_1 \in (0, 5),\, x_2 \in (0, 5) (right)$

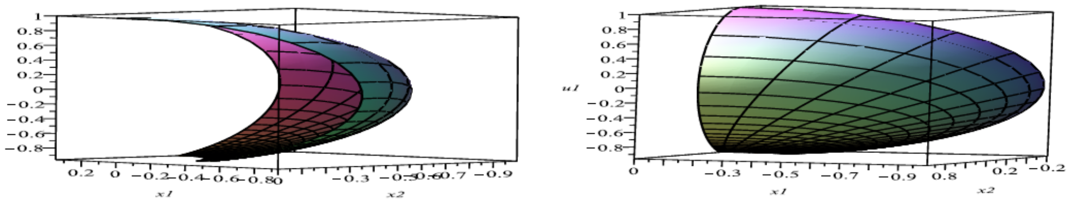

Fig. 9. $t = 2\,(left)\,and\,t = 10\,(right)$

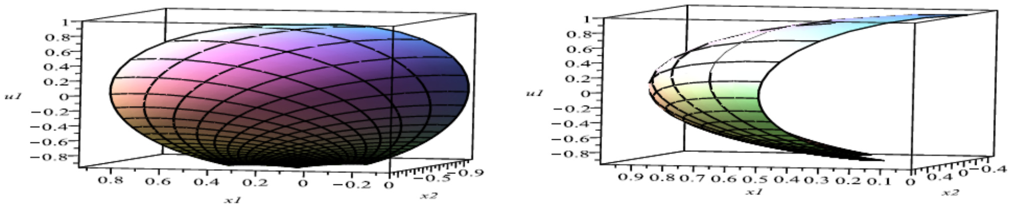

Fig. 10. $t = 20\,(left)\,and\,t = 50,\,(right)$

From above figures we can find that the intensity of magnetization v_1 approaches the south pole $(0, 0, -1)$ from the north pole $(0, 0, 1)$ when the time t tend to ∞.

And then, we study the different cases when x_1, $x_2 \in (0, 0.1)$, $(0, 5)$, $(0, 50)\,and\,(0, 1000)$ and assume $a = \frac{\sqrt{2}}{2}$, $b = \frac{\sqrt{2}}{2}$, $t = 2$, $h = 1$, the solution can be wrote: $v_1 = \frac{1}{1+(x_1^2+x_2^2)}\big((\sqrt{2}x_1 + \sqrt{2}x_2)\cos 2 + (-\sqrt{2}x_1 + \sqrt{2}x_2)\sin 2, -(\sqrt{2}x_1 + \sqrt{2}x_2)\sin 2 + (-\sqrt{2}x_1 + \sqrt{2}x_2)\cos 2, 1 - ({x_1}^2 + {x_2}^2)\big)$

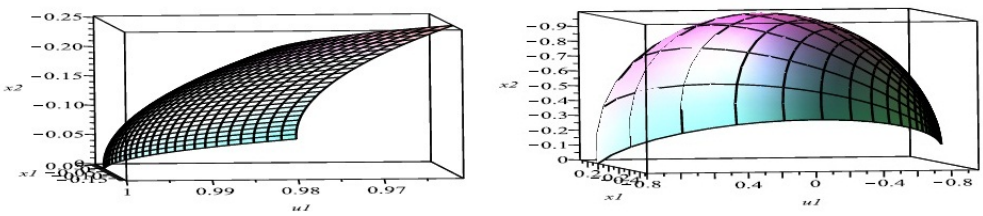

Fig. 11. $x_1 \in (0, 0.1)$, $x_2 \in (0, 0.1)\,(left)\,and\,x_1 \in (0, 5)$, $x_2 \in (0, 5)(right)$

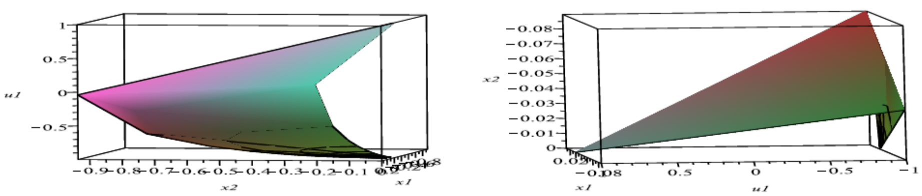

Fig. 12. $x_1 \in (0, 50)$, $x_2 \in (0, 50)\,(left)\,and\,x_1 \in (0, 1000)$, $x_2 \in (0, 1000)(right)$

From above figures we can find that the intensity of magnetization v_1 approaches the south pole $(0, 0, -1)$ from the north pole $(0, 0, 1)$ when the model $|x_1|$ tend to ∞ at the same time.

All in all, we discover the law which when the intensity of magnetization v_1 is moving along one direction, the intensity of magnetization v_1 approaches the south pole $(0, 0, -1)$ from the north pole $(0, 0, 1)$ with the model $|x_1|$ from 0 tend to ∞.

5 Conclusions

a In the section (2), people construct solution of Landau Lifshitz equation usually by hirota method and Backlund transformation. However, in this paper, we construct another style solution that a type of plane wave solution of Landau-Lifshitz equation with the model $|u| = 1$ by different method.

b In the section (4), we discover the law that when the external magnetic field u is moving along one direction, the magnetic field strength u approaches the south pole (0,0,-1) from the north pole (0,0,1) with the model $|x|$ from 0 tend to ∞.

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Competing Interests

Authors have declared that no competing interests exist.

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