British Journal of Mathematics & Computer Science
9(4): 288-299, 2015, Article no.BJMCS.2015.203
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Positive Solutions for a Coupled Nonlinear Fractional Differential System with Coefficients that Change Signs

Zhoujin Cui[1] and Zuodong Yang[2*]

[1] *Jiangsu Maritime Institute, Jiangsu Nanjing 211100, China.*
[2] *School of Teacher Education, Nanjing Normal University, Jiangsu Nanjing 210097, China and Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Jiangsu Nanjing 210023, China.*

Article Information

DOI: 10.9734/BJMCS/2015/17028
Editor(s):
(1) Jacek Dziok, Institute of Mathematics, University of Rzeszow, Poland.
Reviewers:
(1) Nityanand P. Pai, Dept of Mathematics, Manipal University, India.
(2) Hasibun Naher, Department of Mathematics and Natural Sciences, BRAC University, Bangladesh.
(3) Anonymous, India.
Complete Peer review History:
http://www.sciencedomain.org/review-history.php?iid=1145&id=6&aid=9492

Original Research Article

Received: 23 February 2015
Accepted: 04 April 2015
Published: 29 May 2015

Abstract

This paper investigates the existence of positive solutions of the nonlinear fractional differential system

$$\begin{cases} D^s u = \lambda a(t) f(v), & 0 < t < 1, \\ D^p v = \mu b(t) g(u), & 0 < t < 1, \end{cases}$$

where $0 < s, p < 1$, D^s, D^p are the standard Riemann-Liouville fractional derivatives, $\lambda, \mu > 0$ are parameters. The peculiarity of this coupled equations is the coefficient functions $a(t)$ and $b(t)$ change signs, unlike the works in the literature keeping the signs of $a(t), b(t)$ unchanged. On the basis of a nonlinear alternative of Leray-Schauder type and Krasnoselskii′s in a cone, sufficient conditions on $a(t), b(t)$ guarantee the existence of positive solution of the coupled equations are obtained. The results are illustrated with an example.

Keywords: Riemann-Liouville fractional derivatives; Positive solutions; Fixed point theorem in cones.

2010 Mathematics Subject Classification: 47H10; 47H07

**Corresponding author: E-mail: zdyang_jin@263.net*

1 Introduction

The last two decades have witnessed a great progress in fractional calculus and fractional-order dynamical systems. It has been found that fractional calculus is a mathematical tool that works adequately for anomalous social and physical systems with non-local, frequency- and history-dependent properties, and for intermediate states such as soft materials, which are neither idea solid nor idea fluid [1]-[9],[10]. Differential equations with fractional-order derivatives/integrals are called fractional differential equations, and they have found many successful applications in viscoelasticity, heat conduction, electromagnetic wave, diffusion wave, control theory and so on see [11]-[14], and the references therein). Except for a few special cases, it is impossible to find a closed-form solution for a fractional differential equation. Therefore, conditions that govern the existence of some kind of solutions are very important in understanding real systems described by fractional differential equations. In many applications, only the positive solutions of a differential equation admit physical meaning. Thus, a number of works have been made for the existence of positive solutions of fractional differential equations.

For example, Zhang in [15] investigated the existence of a positive solution of the initial problem for the nonlinear fractional equation

$$D^s u = f(t, u), 0 < t < 1, \tag{1}$$

where $0 < s < 1$, D^s is the Riemann-Liouville fractional derivative, and $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is a given continuous function, by using the sub- and supersolution method. Babakhani and Daftardar-Gejji [14] presented a detailed analysis of the existence of positive solutions of multi-term differential equation: $L(D)u = f(x, u)$, where

$$L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}, \ \ 0 < s_1 < \cdots < s_n < 1, \ \ a_j > 0.$$

In [16], [17], Bai et al. studied the following nonlinear fractional differential equation

$$D^s u = \lambda a(t)h(u), \quad 0 < t < 1, \tag{2}$$

and nonlinear fractional differential system

$$\begin{cases} D^s u = f(t, v), & 0 < t < 1, \\ D^p v = g(t, u), & 0 < t < 1, \end{cases} \tag{3}$$

where f, g are two given continuous functions and singular at $t = 0$. Existence of positive solutions for these two problems is established, by means of a nonlinear alternative of Leray-Schauder type and Krasnoselskii′s fixed point theorem in a cone.

As for boundary value problem, El-Shahed [18] discussed the following nonlinear fractional problem

$$\begin{cases} D^s_{0+} u(t) + \lambda a(t)f(u(t)) = 0, & 0 < t < 1, \quad 2 < s \leq 3, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{4}$$

They used the Krasnoselskii′s fixed point theorem on cone expansion and compression to show the existence and non-existence of positive solutions.

Most of the works mentioned above assume that the coefficient functions have a definite sign. To the best of the author's knowledge, little results are available in the literature for the existence of fractional differential equations with coefficients that change signs. This motivates us to study the existence of positive solutions to the following coupled system

$$\begin{cases} D^s u = \lambda a(t)f(v), & 0 < t < 1, \\ D^p v = \mu b(t)g(u), & 0 < t < 1, \end{cases} \tag{5}$$

where $0 < s, p < 1$, D^s, D^p are the standard Riemann-Liouville fractional derivatives, $f, g : [0, \infty) \to [0, \infty)$ are two given continuous functions, $f(0), g(0) > 0$, $a(t), b(t) : [0, 1] \to (-\infty, +\infty)$ may change signs, and $\lambda, \mu > 0$ are parameters.

Schauder's fixed point theorem works well in establishing the existence of a solution of various kind of differential equations, but it does not guarantee the positivity of the solution. The main objective of this paper is to establish two theorems that ensure the existence of positive solution of Eq. (5), by using the following Krasnoselskii's fixed point theorem (Lemma 1.1) and a nonlinear alternative of Leray-Schauder type in cone (Lemma 1.2).

Lemma 1.1 [19] Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and let $A : K \cap (\overline{\Omega}_1 \setminus \Omega_2) \to K$ be continuous and completely continuous. In addition suppose either

(i) $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$, or
(ii)$\|Au\| \ge \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_1 \setminus \Omega_2)$.

Lemma 1.2 [20], [21]. Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a continuous, compact map. Then either

(i) A has a fixed point in $\overline{U}$; or
(ii) there exists $u \in \partial U$ and $\nu \in (0, 1)$ with $u = \nu Au$.

2 Existence of positive solutions

Let $C[0, 1]$ be the space of all continuous real functions defined on $[0, 1]$, $X = C[0, 1] \times C[0, 1]$ be the Banach space endowed with the norm as follows

$$||(u, v)|| = \min\{||u||, ||v||\}, \quad \forall\ (u, v) \in X$$

where $||w|| = \max_{t\in[0,1]} |w(t)|$, $w \in C[0, 1]$. A cone $K \subset X$ is defined by

$$K = \{(u, v) \in X : u(t) \ge 0, v(t) \ge 0, 0 \le t \le 1\}.$$

Definition 2.1[5], [9]. The Riemann-Liouville fractional derivative of order $0 < s < 1$ of a continuous $w : (0, 1) \to R$ is defined to be

$$D^s w(t) = \frac{1}{\Gamma(1-s)} \frac{d}{dt} \int_0^t (t-\tau)^{-s} \omega(\tau) d\tau$$

provided that the right side is point-wise defined on (0, 1).

Definition 2.2. For $0 < s, p < 1$, a pair of $(u, v) \in X$, with continuous fractional derivatives D^s, D^p on $(0, 1)$, is a solution of a coupled system of fractional differential equations (5) if

$$\begin{cases} D^s u(t) = \lambda a(t) f(v(t)), & 0 < t < 1, \\ D^p v(t) = \mu b(t) g(u(t)), & 0 < t < 1, \end{cases} \tag{6}$$

for $s, p \in (0, 1)$ with initial data $u(0) = v(0) = 0$.

The positive solution of Eq. (5) is a solution satisfying $u(0) = v(0) = 0$, $u(t) > 0, v(t) > 0$, $0 < t \le 1$, $(u, v) \in X$, or simply denoted by

$$(u, v) \in D := \{(u, v) \in X :\ u(0) = v(0) = 0,\ u(t) > 0,\ v(t) > 0,\ 0 < t \le 1\}.$$

We assume that

(H_1) $f, g : [0, +\infty) \to [0, +\infty)$ are continuous, nondecreasing functions and $f(0), g(0) > 0$.
(H_2) $a, b : [0, 1] \to R$ are continuous, $a(0) \ne 0, b(0) \ne 0$, and there is $h, l > 1$ such that

$$\int_0^t (t-\tau)^{s-1} a^+(\tau)d\tau \ge h \int_0^t (t-\tau)^{s-1} a^-(\tau)d\tau,$$
$$\int_0^t (t-\tau)^{p-1} b^+(\tau)d\tau \ge l \int_0^t (t-\tau)^{p-1} b^-(\tau)d\tau$$

for $t \in (0, 1]$, where $a^+(t) = \max\{0, a(t)\}$ and $a^-(t) = \max\{0, -a(t)\}$, $b^+(t), b^-(t)$ defined analogously.

According to [22], Proposition 2.4, system (6) is equivalent to the following coupled system of integral equations

$$\begin{cases} u(t) = I^s \lambda a(t) f(v(t)) = \frac{\lambda}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} a(\tau) f(v(\tau))d\tau, \\ v(t) = I^p \mu b(t) g(u(t)) = \frac{\mu}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} b(\tau) g(u(\tau))d\tau, \end{cases} \tag{7}$$

where $t \in [0, 1]$. Define

$$m(t) = \int_0^t (t-\tau)^{s-1} a^+(\tau)d\tau,\ \ q(t) = \int_0^t (t-\tau)^{p-1} b^+(\tau)d\tau.$$

Lemma 2.3. Suppose that $(H_1) - (H_2)$ are satisfied. Then for $0 < \delta_1, \delta_2 < 1$, there exist positive numbers $\overline{\lambda}, \overline{\mu}$ such that, for $0 < \lambda < \overline{\lambda}$, $0 < \mu < \overline{\mu}$, the nonlinear fractional differential equation

$$\begin{cases} D^s u(t) = \lambda a^+(t) f(v(t)), & 0 < t < 1, \\ D^p v(t) = \mu b^+(t) g(u(t)), & 0 < t < 1, \\ u(0) = v(0) = 0 \end{cases} \tag{8}$$

has a positive solution $(\overline{u}_\lambda, \overline{v}_\mu)$ with $||\overline{u}_\lambda||, ||\overline{v}_\mu|| \to 0$ as $\lambda, \mu \to 0$ and

$$\overline{u}_\lambda \ge \lambda \delta_1 f(0) m(t)/\Gamma(s),\quad \overline{v}_\mu \ge \mu \delta_2 g(0) q(t)/\Gamma(p).$$

Proof. It is easy to know from (H_2) that $m(t), q(t) > 0$ for $t \in (0, 1]$. Let $A : X \to X$ be the operator defined as

$$\begin{aligned} A(u, v) = (&\frac{\lambda}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} a^+(\tau) f(v(\tau))d\tau, \frac{\mu}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} b^+(\tau) g(u(\tau))d\tau) \\ &:= (A_1 v(t), A_2 u(t)), \end{aligned} \tag{9}$$

where $u, v \in X$. So the fixed points of A are solutions of system (8). By Lemma 2.1 in [15], $A : K \to K$ is completely continuous. We shall apply the nonlinear alternative of Leray-Schauder type to prove A has at least one fixed point for small λ, μ.
Let $\varepsilon_1, \varepsilon_2 > 0$ be such that

$$f(v) \ge \delta_1 f(0), g(u) \ge \delta_2 g(0), \quad \text{for } 0 \le v \le \varepsilon_1,\ 0 \le u \le \varepsilon_2. \tag{10}$$

Suppose that

$$0 < \lambda < \frac{\Gamma(s)\varepsilon_1}{2||m||f(\varepsilon_1)} := \overline{\lambda}, \ \ 0 < \mu < \frac{\Gamma(p)\varepsilon_2}{2||q||g(\varepsilon_2)} := \overline{\mu}.$$

Since

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty, \quad \lim_{t\to 0^+} \frac{g(t)}{t} = +\infty,$$

and again

$$\frac{f(\varepsilon_1)}{\varepsilon_1} \le \frac{\Gamma(s)}{2\lambda||m||}, \quad \frac{g(\varepsilon_2)}{\varepsilon_2} \le \frac{\Gamma(p)}{2\mu||q||},$$

there exist $R_\lambda \in (0, \varepsilon_1), R_\mu \in (0, \varepsilon_2)$ such that

$$\frac{f(R_\lambda)}{R_\lambda} = \frac{\Gamma(s)}{2\lambda||m||}, \quad \frac{g(R_\mu)}{R_\mu} = \frac{\Gamma(p)}{2\mu||q||}.$$

Let $(u, v) \in K$ be any solution of

$$(u, v) = \nu A(u, v), \tag{11}$$

for each $\nu \in (0, 1)$, where A is given by (9). In fact,

$$\begin{aligned} u(t) &= \frac{\nu\lambda}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} a^+(\tau) f(v(\tau)) d\tau \le \frac{\lambda f(||v||)}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} a^+(\tau) d\tau \\ &= \frac{\lambda f(||(u,v)||)}{\Gamma(s)} m(t) \le \frac{\lambda f(||(u,v)||)}{\Gamma(s)} ||m||, \\ v(t) &= \frac{\nu\mu}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} b^+(\tau) g(u(\tau)) d\tau \le \frac{\mu g(||u||)}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} b^+(\tau) d\tau \\ &= \frac{\mu g(||(u,v)||)}{\Gamma(p)} q(t) \le \frac{\mu g(||(u,v)||)}{\Gamma(p)} ||q||, \end{aligned}$$

Consequently

$$||u|| \le \frac{\lambda f(||(u,v)||)}{\Gamma(s)} ||m||, \ \ ||v|| \le \frac{\mu g(||(u,v)||)}{\Gamma(p)} ||q||. \tag{12}$$

By (12) we can get the following inequalities exist,

$$\frac{f(||(u,v)||)}{||(u,v)||} > \frac{\Gamma(s)}{2\lambda||m||}, \quad \frac{g(||(u,v)||)}{||(u,v)||} > \frac{\Gamma(p)}{2\mu||q||},$$

which implies that $||(u, v)|| \ne R_\lambda$ and $||(u, v)|| \ne R_\mu$. Thus any solution (u, v) of (11) satisfies $||(u, v)|| \ne R_\lambda$ and $||(u, v)|| \ne R_\mu$. Let

$$U = \{(u, v) \in K : ||(u, v)|| < \min\{R_\lambda, R_\mu\}\}.$$

Therefore, Lemma 1.2 guarantees that A has a fixed point $(u, v) \in \overline{U} \cap D$. Moreover, combining (9) and (10), we obtain

$$\overline{u}_\lambda \geq \lambda\delta_1 f(0)m(t)/\Gamma(s), \quad \overline{v}_\mu \geq \mu\delta_2 g(0)q(t)/\Gamma(p).$$

Hence system (8) has a positive solution.

Theorem 2.4. Suppose that $(H_1)-(H_2)$ hold. Then there exist positive numbers λ^*, μ^* such that system (6) has at least one positive solution for $\lambda \in (0, \lambda^*), \mu \in (0, \mu^*)$.

Proof. Let

$$m^*(t) = \int_0^t (t-\tau)^{s-1}a^-(\tau)d\tau, \quad q^*(t) = \int_0^t (t-\tau)^{p-1}b^-(\tau)d\tau. \tag{13}$$

Then $m^*(t), q^*(t) \geq 0$ for each $t \in (0, 1]$. By (H_2) we have $m(t) \geq hm^*(t)$, $q(t) \geq lq^*(t)$. Choose $0 < c < 1$ such that $hc, lc > 1$. There is $b > 0$ such that $f(x) \leq hcf(0), g(x) \leq lcg(0)$ for $x \in [0, b]$, then

$$m^*(t)f(x) \leq cm(t)f(0), q^*(t)g(x) \leq cq(t)g(0) \quad \text{for } t \in (0, 1], \ x \in [0, b]. \tag{14}$$

Fix $\delta_1, \delta_2 \in (c, 1)$ and let $\lambda^*, \mu^* > 0$ be such that

$$||\overline{u}_\lambda|| + \frac{\lambda\delta_1 f(0)||m||}{\Gamma(s)} \leq b, \ ||\overline{v}_\mu|| + \frac{\mu\delta_2 g(0)||q||}{\Gamma(p)} \leq b, \ \lambda \in (0, \lambda^*), \mu \in (0, \mu^*), \tag{15}$$

in which $\overline{u}_\lambda, \overline{v}_\mu$ are given by Lemma 2.3, and

$$|f(x) - f(y)| \leq f(0)\frac{\delta_1 - c}{2}, \ |g(x) - g(y)| \leq g(0)\frac{\delta_2 - c}{2}, \tag{16}$$

for $x, y \in [0, b]$ with

$$|x - y| \leq \min\{\frac{\lambda^*\delta_1 f(0)||m||}{\Gamma(s)}, \frac{\mu^*\delta_2 g(0)||q||}{\Gamma(p)}\}.$$

Let $\lambda \in (0, \lambda^*), \mu \in (0, \mu^*)$. We look for a solution (u_λ, v_μ) of the form $(\overline{u}_\lambda + u^*_\lambda, \overline{v}_\mu + v^*_\mu)$, where $(\overline{u}_\lambda, \overline{v}_\mu)$ is the solution of (8) given by Lemma 2.1. Thus (u^*_λ, v^*_μ) solves the following equation:

$$\begin{cases} D^s u^*_\lambda(t) = \lambda a^+(t)[f(\overline{v}_\mu + v^*_\mu) - f(\overline{v}_\mu)] - \lambda a^-(t)f(\overline{v}_\mu + v^*_\mu), & 0 < t < 1, \\ D^p v^*_\mu(t) = \mu b^+(t)[g(\overline{u}_\lambda + u^*_\lambda) - g(\overline{u}_\lambda)] - \mu b^-(t)g(\overline{u}_\lambda + u^*_\lambda), & 0 < t < 1, \\ u^*_\lambda(0) = v^*_\mu(0) = 0. \end{cases}$$

For each $(\overline{u}, \overline{v}) \in X$, let $(w, z) = A(\overline{u}, \overline{v})$ be the solution of

$$\begin{cases} D^s w = \lambda a^+(t)[f(\overline{v}_\mu + \overline{v}) - f(\overline{v}_\mu)] - \lambda a^-(t)f(\overline{v}_\mu + \overline{v}), & 0 < t < 1, \\ D^p z = \mu b^+(t)[g(\overline{u}_\lambda + \overline{u}) - g(\overline{u}_\lambda)] - \mu b^-(t)g(\overline{u}_\lambda + \overline{u}), & 0 < t < 1, \\ w(0) = z(0) = 0. \end{cases}$$

Then $A : X \to X$ is completely continuous. Let $(\overline{u}, \overline{v}) \in X$ and $\nu \in (0, 1)$ be such that $(\overline{u}, \overline{v}) = \nu A(\overline{u}, \overline{v})$, then we have

$$\begin{cases} D^s \overline{u} = \lambda a^+(t)[f(\overline{v}_\mu + \overline{v}) - f(\overline{v}_\mu)] - \lambda a^-(t)f(\overline{v}_\mu + \overline{v}), & 0 < t < 1, \\ D^p \overline{v} = \mu b^+(t)[g(\overline{u}_\lambda + \overline{u}) - g(\overline{u}_\lambda)] - \mu b^-(t)g(\overline{u}_\lambda + \overline{u}), & 0 < t < 1, \end{cases}$$

that is

$$\overline{u} = \frac{\nu\lambda}{\Gamma(s)} \int_0^t (t-\tau)^{s-1}a^+(\tau)[f(\overline{v}_\mu(\tau) + \overline{v}(\tau)) - f(\overline{v}_\mu(\tau))]d\tau$$

$$-\frac{\nu\lambda}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}a^-(\tau)f(\overline{v}_\mu(\tau)+\overline{v}(\tau))d\tau, \tag{17}$$

$$\overline{v} = \frac{\nu\mu}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}b^+(\tau)[g(\overline{u}_\lambda(\tau)+\overline{u}(\tau))-g(\overline{u}_\lambda(\tau))]d\tau$$

$$-\frac{\nu\mu}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}b^-(\tau)g(\overline{u}_\lambda(\tau)+\overline{u}(\tau))d\tau. \tag{18}$$

We claim that $||\overline{u}|| \neq \lambda\delta_1 f(0)||m||/\Gamma(s), ||\overline{v}|| \neq \mu\delta_2 g(0)||q||/\Gamma(p)$. Suppose to the contrary that $(||\overline{u}||, ||\overline{v}||) = (\lambda\delta_1 f(0)||m||/\Gamma(s), \mu\delta_2 g(0)||q||/\Gamma(p))$. Then by (15) and (16), we get

$$||\overline{v}_\mu+\overline{v}|| \leq ||\overline{v}_\mu|| + ||\overline{v}|| \leq b,\ \ ||\overline{u}_\lambda+\overline{u}|| \leq ||\overline{u}_\lambda|| + ||\overline{u}|| \leq b \tag{19}$$

and

$$|f(\overline{v}_\mu+\overline{v}) - f(\overline{v}_\mu)| \leq f(0)\frac{\delta_1-c}{2},\ \ |g(\overline{u}_\lambda+\overline{u}) - g(\overline{u}_\lambda)| \leq g(0)\frac{\delta_2-c}{2}. \tag{20}$$

From (14), we get

$$m^*(t)f(b) \leq cm(t)f(0), q^*(t)g(b) \leq cq(t)g(0) \quad \text{for } t \in (0,1]. \tag{21}$$

Using (17)-(21), we obtain for each $t \in (0,1]$ that

$$\begin{aligned}
\overline{u} &\leq \frac{\lambda}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}a^+(\tau)[f(\overline{v}_\mu(\tau)+\overline{v}(\tau))-f(\overline{v}_\mu(\tau))]d\tau \\
&+ \frac{\lambda}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}a^-(\tau)f(\overline{v}_\mu(\tau)+\overline{v}(\tau))d\tau \\
&\leq \frac{\lambda}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}a^+(\tau)f(0)\frac{\delta_1-c}{2}d\tau + \frac{\lambda f(b)}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}a^-(\tau)d\tau \\
&\leq \frac{\lambda(\delta_1-c)f(0)}{2\Gamma(s)}m(t) + \frac{\lambda c f(0)}{\Gamma(s)}m(t) \\
&= \frac{\lambda(\delta_1+c)f(0)}{2\Gamma(s)}m(t).
\end{aligned} \tag{22}$$

Similarly we can get

$$|\overline{v}| \leq \frac{\mu(\delta_2+c)g(0)}{2\Gamma(p)}q(t). \tag{23}$$

In particular

$$||\overline{u}|| \leq \frac{\lambda(\delta_1+c)f(0)}{2\Gamma(s)}||m|| < \frac{\lambda\delta_1 f(0)}{\Gamma(s)}||m||,\ \ ||\overline{v}|| \leq \frac{\mu(\delta_2+c)g(0)}{2\Gamma(p)}||q|| < \frac{\mu\delta_2 g(0)}{\Gamma(p)}||q||,$$

a contradiction, and the claim is proved. Let

$$U = \{u \in X : ||(u,v)|| < \min\{\frac{\lambda\delta_1 f(0)}{\Gamma(s)}||m||, \frac{\mu\delta_2 g(0)}{\Gamma(p)}||q||\}\}.$$

By Lemma 1.2, A has a fixed point $(u^*_\lambda, v^*_\mu) \in \overline{U}$. Consequently,

$$||u^*_\lambda|| \leq \frac{\lambda\delta_1 f(0)}{\Gamma(s)}||m||,\ \ ||v^*_\mu|| \leq \frac{\mu\delta_2 g(0)}{\Gamma(p)}||q||.$$

Hence (u^*_λ, v^*_μ) satisfies (22), (23) and, using Lemma 2.3, we get

$$u_\lambda \geq \overline{u}_\lambda - |u^*_\lambda| \geq \frac{\lambda\delta_1 f(0)}{\Gamma(s)}m(t) - \frac{\lambda(\delta_1+c)f(0)}{2\Gamma(s)}m(t) = \frac{\lambda(\delta_1-c)f(0)}{2\Gamma(s)}m(t),$$

$$v_\mu \geq \overline{v}_\mu - |v_\mu^*| \geq \frac{\mu\delta_2 g(0)}{\Gamma(p)}q(t) - \frac{\mu(\delta_2+c)g(0)}{2\Gamma(p)}q(t) = \frac{\mu(\delta_2-c)g(0)}{2\Gamma(p)}q(t),$$

i.e., (u_λ, v_μ) is a positive solution of system (6). So the proof of Theorem 2.4 is complete.

Theorem 2.5. Let $0 < \lambda^1 < \lambda < \lambda^*, 0 < \mu^1 < \mu < \mu^*$, $0 < \delta_1 < s < 1$, and $0 < \delta_2 < p < 1$. Let $t^{\delta_1} f(y)$ and $t^{\delta_2} g(y)$ are two continuous functions on $[0,1]\times[0,\infty)$. $a, b : [0,1] \to (0,+\infty)$ are continuous and there exist two distinct constants $0 < c < C$ such that $c \leq a, b \leq C$. Assume that there exist two distinct positive constants $\rho, \eta, \rho > \eta$, such that the following conditions are satisfied:

$$(H_1') \;\; t^{\delta_1} f(\omega) \leq \rho\frac{\Gamma(1-\delta_1+s)}{C\lambda^*\Gamma(1-\delta_1)} \;\text{ and }\; t^{\delta_2} g(\omega) \leq \rho\frac{\Gamma(1-\delta_2+p)}{C\mu^*\Gamma(1-\delta_2)}, \quad (t,\omega)\in[0,1]\times[0,\rho],$$
$$t^{\delta_1} f(\omega) \geq \eta\frac{\Gamma(1-\delta_1+s)}{c\lambda^1\Gamma(1-\delta_1)} \;\text{ and }\; t^{\delta_2} g(\omega) \geq \eta\frac{\Gamma(1-\delta_2+p)}{c\mu^1\Gamma(1-\delta_2)}, \quad (t,\omega)\in[0,1]\times[0,\eta],$$

Then system (6) has at least one positive solution.

Proof. Let Let $A : X \to X$ be the operator defined as

$$A(u,v) = (\frac{\lambda}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}a(\tau)f(v(\tau))d\tau, \frac{\mu}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}b(\tau)g(u(\tau))d\tau)$$
$$:= (A_1 v(t), A_2 u(t)), \tag{24}$$

where $u, v \in X$. By Lemma 2.1 in [15], $A : K \to D$ is completely continuous. In order to apply Lemma 1.1, we separate the proof into the following two steps.

Step 1. Let $U_2 = \{(u,v) \in K : ||(u,v)|| \leq \rho\}$. For $(u,v) \in K \cap \partial U_2$, we have $0 \leq u(t) \leq \rho, 0 \leq v(t) < \rho$ for all $t \in [0,1]$. By assumption (H_1') that for $t \in [0,1]$,

$$A_1 v(t) = \frac{\lambda}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}a(\tau)\tau^{-\delta_1}\tau^{\delta_1}f(v(\tau))d\tau$$
$$\leq \rho\frac{\Gamma(1-\delta_1+s)}{\Gamma(1-\delta_1)}\frac{1}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}\tau^{-\delta_1}d\tau = \rho\frac{\Gamma(1-\delta_1+s)}{\Gamma(1-\delta_1)}\frac{\Gamma(1-\delta_1)}{\Gamma(1-\delta_1+s)}t^{s-\delta_1} = \rho t^{s-\delta_1}$$

and

$$A_2 u(t) = \frac{\mu}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}b(\tau)g(u(\tau))d\tau \leq \rho t^{p-\delta_2}.$$

Hence for $(u,v) \in K \cap \partial U_2$,

$$||A(u,v)|| = \max\{\max_{0\leq t\leq 1}|A_1 v(t)|, \max_{0\leq t\leq 1}|A_2 u(t)|\} \leq \rho = ||(u,v)||.$$

Step 2. Let $U_1 = \{(u,v) \in K : ||(u,v)|| < \eta\}$. For $(u,v) \in K \cap \partial U_1$, we have $0 \leq u(t) \leq \eta, 0 \leq v(t) \leq \eta$ for all $t \in [0,1]$. It follows from (H_1') that for $t \in [0,1]$,

$$A_1 v(1) = \frac{\lambda}{\Gamma(s)}\int_0^1 (1-\tau)^{s-1}a(\tau)\tau^{-\delta_1}\tau^{\delta_1}f(v(\tau))d\tau$$
$$\geq \eta\frac{\Gamma(1-\delta_1+s)}{\Gamma(1-\delta_1)}\frac{1}{\Gamma(s)}\int_0^1 (1-\tau)^{s-1}\tau^{-\delta_1}d\tau = \eta$$

and

$$A_2 u(1) = \frac{\mu}{\Gamma(p)}\int_0^1 (1-\tau)^{p-1}b(\tau)g(u(\tau))d\tau \geq \eta.$$

Hence for $(u, v) \in K \cap \partial U_1$,

$$||A(u, v)|| = \max\{\max_{0\le t\le 1} |A_1 v(t)|, \max_{0\le t\le 1} |A_2 u(t)|\} \ge \eta = ||(u, v)||.$$

Therefore, by (ii) of Lemma 1.1, we complete the proof.

3 Example

Finally, we give an example to illustrate the result obtained in this paper.

Example 3.1. Consider the following nonlinear fractional differential system:

$$\begin{cases} D^s u(t) = \lambda(2-3t)(v^2 + e^v + \frac{1}{5}), \quad 0 < t < 1, \\ D^p v(t) = \mu(\frac{3}{4} - t)(u^4 + \ln(1+u) + \frac{1}{3}), \quad 0 < t < 1, \\ u(0) = v(0) = 0. \end{cases} \tag{25}$$

in which $1/2 < s < 1, 1/3 < p < 1$ and $\lambda, \mu > 0$ are parameters.

Let $f(v) = v^2 + e^v + \frac{1}{5}, g(u) = u^4 + ln(1+u) + \frac{1}{3}$ and $a(t) = 2 - 3t, b(t) = 3/4 - t$. Obviously, f and g satisfy (H_1). In the following, we verify that a, b satisfy (H_2). We see

$$a^+(t) = \begin{cases} 2-3t, & 0 \le t \le \frac{2}{3}, \\ 0, & \frac{2}{3} < t \le 1, \end{cases} \qquad a^-(t) = \begin{cases} 0, & 0 \le t \le \frac{2}{3}, \\ 3t-2, & \frac{2}{3} < t \le 1. \end{cases}$$

and

$$b^+(t) = \begin{cases} \frac{3}{4} - t, & 0 \le t \le \frac{3}{4}, \\ 0, & \frac{3}{4} < t \le 1, \end{cases} \qquad b^-(t) = \begin{cases} 0, & 0 \le t \le \frac{3}{4}, \\ t - \frac{3}{4}, & \frac{3}{4} < t \le 1. \end{cases}$$

If $\frac{2}{3} < t \le 1$, then

$$\begin{aligned} p(t) &= \int_0^t (t-\tau)^{s-1} a^+(\tau) d\tau = \int_0^{2/3} (t-\tau)^{s-1}(2-3\tau) d\tau \\ &= 2\int_0^{2/3} (t-\tau)^{s-1} d\tau + 3\int_0^{2/3} (t-\tau)^{s-1}[(t-\tau) - t] d\tau \\ &= (2-3t)\int_0^{2/3} (t-\tau)^{s-1} d\tau + 3\int_0^{2/3} (t-\tau)^s d\tau \\ &= \frac{3}{s+1} t^{s+1} - \frac{1}{s}(3t-2)t^s + \frac{3}{s(s+1)}(t - \frac{2}{3})^{s+1}, \end{aligned} \tag{26}$$

and

$$p^*(t) = \int_0^t (t-\tau)^{s-1} a^-(\tau) d\tau = \frac{3}{s(s+1)}(t - \frac{2}{3}) t^{s+1}. \tag{27}$$

For $1/2 < s < 1$, setting

$$\varepsilon := \frac{s - \frac{1}{3}(s+1)}{(\frac{1}{3})^{s+1}} > 0.$$

By $2/3 < t \le 1$, we have $(t - 2/3)/t \le \frac{1}{3}$. Therefore,

$$s = \frac{1}{3}(s+1) + \varepsilon(\frac{1}{3})^{s+1} \ge (s+1)\frac{t - 2/3}{t} + \varepsilon(\frac{t-2/3}{t})^{s+1},$$

that is

$$st^{s+1} \geq (s+1)(t-2/3)t^s + \varepsilon(t-\frac{2}{3})^{s+1}, \quad \frac{2}{3} < t \leq 1. \tag{28}$$

By (26)-(28), we get for each $t \in (2/3, 1]$ that

$$p(t) \geq (1+\varepsilon)p^*(t). \tag{29}$$

If $0 < t \leq 2/3$, then

$$\int_0^t (t-\tau)^{s-1} a^+(\tau)d\tau = \int_0^t (t-\tau)^{s-1}(2-3\tau)d\tau > 0, \tag{30}$$

and

$$\int_0^t (t-\tau)^{s-1} a^-(\tau)d\tau = 0, \tag{31}$$

which implies (29) holds also for any $t \in (0, 2/3]$. Thus, a satisfies (H_2). Similarly, we can get b satisfies (H_2). Applying Theorem 2.4, we know that there are numbers $\lambda^*, \mu^* > 0$ such that system (25) has at least one positive solution.

4 Conclusion

This paper investigates the existence of positive solutions of the nonlinear fractional differential system. The peculiarity of this coupled equations is the coefficient functions $a(t)$ and $b(t)$ change signs, unlike the works in the literature keeping the signs of $a(t), b(t)$ unchanged. On the basis of a nonlinear alternative of Leray-Schauder type and Krasnoselskii′s in a cone, sufficient conditions on $a(t), b(t)$ guarantee the existence of positive solution of the coupled equations are obtained. Further work will focus on the numerical solutions to the specific issues.

Acknowledgment

Project Supported by the National Natural Science Foundation of China(No.11171092 and No.11471164); A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions(PAPD).

Competing Interests

The authors declare that no competing interests exist.

References

[1] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equation. I Wiley, NewYork; 1993.

[2] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives. theory and Applications. Gordon and Breach, London; 1993.

[3] Kiryakova V. Generalized fractional calculus and applications, in: pitman research notes mathematical series. Longman, Harlow. 1994;301.

[4] Diethelm K, Freed AD. On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity, in Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties. Springer-Verlag, Heidelberg. 1999;217-224.

[5] Podlubny I. Fractional differential equations, vol. 198 of mathematics in science and engineering. Academic Press, San Diego, Calif, USA; 1999.

[6] Hilfer R. Applications of fractional calculus in Physics. World Scientific, Singapore; 2000.

[7] Agrawal OP. Formulation of Euler-Larange equations for fractional variational problems. J. Math. Anal. Appl. 2002;272:368-379.

[8] Xu MY, Tan W. Intermediate processes and critical phenomena: Theory, method and progress of fractional operators and their applications to modern mechanics. Science in China (Series G). 2006;49(3):257-272.

[9] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Vol. 204 of North-Holland Mathematics Studies, Elsevier Science; 2006. Amsterdam, The Netherlands.

[10] Liang J, Liu Z, Wang X. Solvability for a couple system of nonlinear fractional differential equations in a Banach space, Fract. Calc. Appl. Anal. 2013;16:51-63.

[11] Kilbas AA, Trujillo JJ. Computation of fractional integrals via functions of hypergeometric and Bessel type. J. Comput. Appl. Math. 2000;118:223-239.

[12] He JH. Nonlinear oscillation with fractional derivative and its applications. International Conference on Vibration Engineering'98, Dalian, China. 1998;288-291.

[13] Atanackovic TM, Stankovic B. On a system of differential equations with fractional derivatives arising in rod theory. J. Phys. A. 2004;37: 1241-1250.

[14] Babakhani A, Daftardar-Gejji V. Existence of positive solutions of nonlinear fractional differential equations. J. Math. Anal. Appl. 2003;278:434-442.

[15] Zhang S. The existence of a positive solution for a nonlinear fractional differential equation. J. Math. Anal. Appl. 2000;252:804-812.

[16] Bai C. Positive solutions for nonlinear fractional differential equations with coefficient that changes sign. Nonlinear Anal. 2006;64:677-685.

[17] Bai C, Fang J. The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations. Appl. Math. Comput. 2004;150:611-621.

[18] El-Shahed M. Positive solutions for boundary value problem of nonlinear fractional differential equation. Abstract and Applied Analysis; 2007. Article ID 10368, 8 pages. DOI:10.1155/2007/10368.

[19] Krasnoselskii MA. Positive solutions of operator equations. Noordhoff, Groningen; 1964.

[20] Agarwal RP, Meehan M, O'Regan D. Fixed point theory and applications. Cambridge University Press, Cambridge; 2001.

[21] Granas A, Guenther RB, Lee JW. Some general existence principle in the Caratheo-dory theory of nonlinear systems. J. Math. Pures Appl. 1991;70:153-196.

[22] Delbosco D, Rodino L. Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl. 1996;204:609-625.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=1145&id=6&aid=9492