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Coefficients Bounds for Certain Classes of Analytic Functions of Complex Order

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Abstract

In this paper, we determine coefficients bounds for functions in certain subclasses of analytic functions of complex order, which are introduced here by means of the nonhomogeneous Cauchy-Euler differential equation of order m. Our main result contain some corollaries as special cases.

Keywords: Analytic functions, Coefficient bounds; Starlike functions of complex order; Convex functions of complex order; Nonhomogeneous Cauchy-Euler differential equations

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1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of complex order $\gamma(\gamma \in \mathbb{C}^* := \mathbb{C}\diagdown\{0\})$ and type $\beta(0 \le \beta < 1)$, that is $f(z) \in \mathcal{S}^*_\gamma(\beta)$, if and only if

$$\text{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \beta \qquad (z \in \mathcal{U}; \gamma \in \mathbb{C}^*), \tag{1.2}$$

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and is said to be convex of complex order $\gamma(\gamma \in \mathbb{C}^*)$ and type $\beta(0 \leq \beta < 1)$, denoted by $\mathcal{C}_\gamma(\beta)$ if and only if

$$\text{Re}\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > \beta \qquad (z \in \mathcal{U}; \gamma \in \mathbb{C}^*). \tag{1.3}$$

The classes $\mathcal{S}^*_\gamma(\beta)$ and $\mathcal{C}_\gamma(\beta)$ were introduced by the first author in [1]. Note that $\mathcal{S}^*_\gamma(0) = \mathcal{S}^*_\gamma$ and $\mathcal{C}_\gamma(0) = \mathcal{C}_\gamma$ the classes considered earlier by Nasr and Aouf [2] and Wiatrowski [3]. Also, $\mathcal{S}^*_1(\beta) = \mathcal{S}^*(\beta)$ and $\mathcal{C}_1(\beta) = \mathcal{C}(\beta)$ which are, respectively, the familiar classes of starlike functions of order $\beta(0 \leq \beta < 1)$ and convex functions of order $\beta(0 \leq \beta < 1)$.

Let $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition

$$\text{Re}\left[1 + \frac{1}{\gamma}\left(\frac{z[\lambda\mu z^2 f''(z) + (\lambda-\mu)zf'(z) + (1-\lambda+\mu)f(z)]'}{\lambda\mu z^2 f''(z) + (\lambda-\mu)zf'(z) + (1-\lambda+\mu)f(z)} - 1\right)\right] > \beta \tag{1.4}$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in \mathbb{C}^*$ and $z \in \mathcal{U}$.

For $\mu = 0$, the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ reduces to the class $\mathcal{SC}(\gamma, \lambda, \beta)$ introduced by Altintaş et al. [4]. Clearly, we have $\mathcal{Q}(\gamma, 0, 0, \beta) = \mathcal{S}^*_\gamma(\beta)$ and $\mathcal{Q}(\gamma, 1, 0, \beta) = \mathcal{C}_\gamma(\beta)$.

In the present paper, we propose to derive some coefficient bounds for the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ and also for functions in the subclass $\mathcal{H}(\gamma, \lambda, \mu, \beta, m; \zeta)$ of $\mathcal{A}$, which consists of functions $f(z) \in \mathcal{A}$ satisfying the following nonhomogeneous Cauchy-Euler differential equation of order m :

$$z^m\frac{d^m w}{dz^m} + \binom{m}{1}(\zeta + m - 1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots + \binom{m}{m}w\prod_{j=0}^{m-1}(\zeta + j) = g(z)\prod_{j=0}^{m-1}(\zeta + j + 1) \tag{1.5}$$

$$(w = f(z); g(z) \in \mathcal{Q}(\gamma, \lambda, \mu, \beta),\ \zeta \in\ \mathbb{R}\backslash(-\infty, -1]; m \in \mathbb{N}^* := \mathbb{N}\backslash\{1\} = \{2, 3, ...\}).$$

2 Coefficient Estimates

We begin by obtaining coefficient bounds for functions in the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$.

Theorem 2.1. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{Q}(\gamma, \lambda, \mu, \beta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|\gamma|(1-\beta)]}{(n-1)![1 + (\lambda\mu n + \lambda - \mu)(n-1)]} \qquad (n \in \mathbb{N}^*), \tag{2.1}$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1$, and $\gamma \in C^*$.

Proof. Let the function $F(z)$ be defined by

$$F(z) = \lambda\mu z^2 f''(z) + (\lambda-\mu)zf'(z) + (1-\lambda+\mu)f(z) \quad (f \in \mathcal{A}; z \in \mathcal{U}). \tag{2.2}$$

Then $F(z)$ is analytic in $\mathcal{U}$ with $F(0) = F'(0) - 1 = 0$. From (1.1) and (2.2) it is easily seen that

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k \qquad (z \in \mathcal{U}).$$

where

$$A_k := [1 + (\lambda\mu k + \lambda - \mu)(k-1)]a_k \qquad (k \in \mathbb{N}^*). \tag{2.3}$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{F(z)} - 1\right) - \beta}{1 - \beta}$$

or, equivalently,

$$zF'(z) - F(z) = \gamma(1 - \beta)(p(z) - 1)F(z) \tag{2.4}$$

then $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathcal{U}$ and Re$\{p(z)\} > 0$. Therefore, we have $|c_n| \leq 2 \quad (n \in \mathbb{N})$. From (2.4), it follows that

$$(n-1)A_n = \gamma(1 - \beta)(c_{n-1} + c_{n-2}A_2 + \cdots + c_1 A_{n-1}).$$

In particular, for $n = 2, 3, 4,$ we have

$$\begin{aligned} |A_2| &\leq 2\,|\gamma|\,(1-\beta), \\ |A_3| &\leq \frac{2\,|\gamma|\,(1-\beta)[1 + 2\,|\gamma|\,(1-\beta)]}{2!}, \end{aligned}$$

and

$$|A_4| \leq \frac{2\,|\gamma|\,(1-\beta)[1 + 2\,|\gamma|\,(1-\beta)][2 + 2\,|\gamma|\,(1-\beta)]}{3!},$$

respectively. Thus, by using the principle of mathematical induction, we obtain

$$|A_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2\,|\gamma|\,(1-\beta)]}{(n-1)!} \qquad (n \in \mathbb{N}^*). \tag{2.5}$$

From (2.3) it is clear that

$$A_n = [1 + (\lambda\mu n + \lambda - \mu)(n-1)]a_n \qquad (n \in \mathbb{N}^*). \tag{2.6}$$

Now the inequality (2.1) follows immediately from (2.5) and (2.6). This completes the proof of Theorem 2.1. □

Putting $\mu = \lambda = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1)and satisfies the condition*

$$\text{Re}\left[1 + \frac{1}{\gamma}\left(\frac{z[z^2 f''(z) + f(z)]'}{z^2 f''(z) + f(z)} - 1\right)\right] > \beta \tag{2.7}$$

then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2\,|\gamma|\,(1-\beta)]}{(n^2 - n + 1)(n-1)!} \qquad (n \in \mathbb{N}^*), \tag{2.8}$$

where $0 \leq \beta < 1,$ and $\gamma \in C^*$.

Putting $\mu = 0$ in Theorem 2.1, we get the following result obtained by Altintaş et al. [5].

Corollary 2.3. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{SC}(\gamma, \lambda, \beta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2\,|\gamma|\,(1-\beta)]}{(n-1)![1 + \lambda(n-1)]} \qquad (n \in \mathbb{N}^*), \tag{2.9}$$

where $0 \leq \lambda \leq 1; 0 \leq \beta < 1,$ *and* $\gamma \in C^*.$

Finally, we prove the following theorem.

Theorem 2.4. *Let the function* $f(z) \in \mathcal{A}$ *be given by (1.1). If* $f(z) \in \mathcal{H}(\gamma, \lambda, \mu, \beta, m; \zeta)$, *then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\gamma|(1-\beta)] \prod_{j=0}^{m-1} (\zeta + j + 1)}{(n-1)![1 + (\lambda\mu n + \lambda - \mu)(n-1)] \prod_{j=0}^{m-1} (\zeta + j + n)} \qquad (m, n \in \mathbb{N}^*), \tag{2.10}$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in C^*$ *and* $\zeta \in \mathbb{R} \backslash (-\infty, -1].$

Proof. Let the function $f(z) \in \mathcal{A}$ be given by (1.1). Also let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{Q}(\gamma, \lambda, \mu, \beta).$$

Then from (1.5), we get

$$a_n = \left(\frac{\prod_{j=0}^{m-1} (\zeta + j + 1)}{\prod_{j=0}^{m-1} (\zeta + j + n)} \right) b_n \qquad (n \in \mathbb{N}^*; \zeta \in \ R \backslash (-\infty, -1]).$$

Thus, by using Theorem 2.1 , we readily obtain the inequality (2.10) . ☐

Putting $\mu = \lambda = 1$ in Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let the function* $f(z) \in \mathcal{A}$ *be given by (1.1). If* $f(z)$ *satisfies the equation (1.5) and* $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ *satisfies the condition (2.7), then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\gamma|(1-\beta)] \prod_{j=0}^{m-1} (\zeta + j + 1)}{(n^2 - n + 1)(n-1)! \prod_{j=0}^{m-1} (\zeta + j + n)} \qquad (m, n \in \mathbb{N}^*), \tag{2.11}$$

where $0 \leq \beta < 1; \gamma \in C^*$ *and* $\zeta \in \mathbb{R} \backslash (-\infty, -1].$

Putting $\mu = 0$ and $m = 2$ in Theorem 2.4, we get the following result obtained by Altintaş et al. [5].

Corollary 2.6. *Let the function* $f(z) \in \mathcal{A}$ *be given by (1.1). If* $f(z)$ *satisfies the nonhomogeneous Cauchy-Euler differential equation of order* $2,$ *given by (1.5) and* $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ *satisfies the condition (2.7), then*

$$|a_n| \leq \frac{(\zeta+1)(\zeta+2) \prod_{j=0}^{n-2} [j + 2|\gamma|(1-\beta)]}{(n-1)![1 + (\lambda\mu n + \lambda - \mu)(n-1)](\zeta + n)(\zeta + n + 1)} \qquad (n \in \mathbb{N}^*), \tag{2.12}$$

where $0 \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in C^*$ *and* $\zeta \in \mathbb{R}\backslash(-\infty, -1]$.

A similar work can also be referred to Eker et al. [6]. In this article they studied the Dziok-Srivastava operator.

Open problem: Is it possible to solve problems related to the Fekete-Szegö theorem as given in [7]? It is yet to be solved.

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Competing Interests

The authors declare that no competing interests exist.

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