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Asymptotic Domain Decomposition and a Posteriori Estimates for a Semi Linear Problem

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Abstract

Coupling heterogeneous mathematical models is today commonly used, and effective solution methods for the resulting hybrid problem have recently become available for several systems. Even if in certain circumstances, asymptotic evaluations of the location of the interfaces are available, no strategy are proposed for locating the interfaces in numerical simulations. In this article, a semi-linear elliptic problem is considered. By reformulating the problem in a mixed formulation context and by using an a posteriori error estimate, we propose an indicator of the error due to a wrong position of the junction. Minimizing this indicator allows us to determine accurately the location of the junction. By comparing this indicator with a mesh error indicator, this allows to decide if it is better to refine the mesh or to move the interface. Some numerical results are presented showing the efficiency of the proposed indicator.

Keywords: Method of asymptotic partial domain decomposition; a posteriori error estimates; Indicator of the error; semi linear elliptic equations.

2010 Mathematics Subject Classification: 35F40; 65

1 Introduction

The method of asymptotic partial decomposition of a domain (MAPDD) originates with the works of G.Panasenko Panasenko (2005). The idea is to replace an original $3D$ or $2D$ problem by an hybrid one $3D-1D$; or $2D-1D$ where the dimension of the problem decreases in part of the domain. The location of the junction between the heterogenous problems is asymptotically estimated in the works of G.Panasenko mainly for linear problems. Nevertheless for numerical simulations it is essential to detect with accuracy the location of the junction. Let us also mention the interest of locating with accuracy the position of the junction in blood flows simulations when different nonlinear mathematical models are used Quarteroni and Veneziani (2003), or in fluid/solid problems for which subproblems

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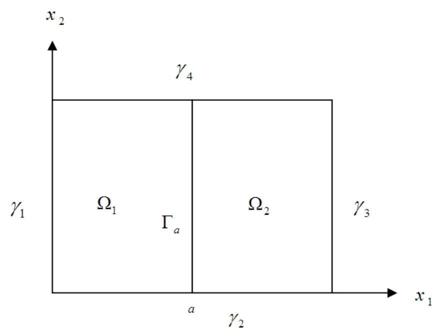
are computed with independent black-box code Blanco et all (2010), Leiva et all (2011). Here the method proposed is to determine the location of the junction (i.e the location of the boundary Γ in the example treated) by using optimization techniques and a posteriori error estimates. In the presented problem due to the specific right hand side f it is assumed that the solution U could be approximated by a 1D solution in a part of the domain. That allows saving computing time when the problem is numerically solved. For implementing such a strategy, we have to find admissible transmission conditions and locating the interface. First it is shown that MAPDD can be expressed with a mixed domain decomposition formulation for a semi linear elliptic problem in two different ways. Then an a posteriori error estimate is derived for locating the best position of the junction. The idea to use a posteriori error estimates for optimization problems have extensively been used, the reader is referred to Becker and R. Rannacher (2003) for example.
In the following, the problem handled, is described, and the introduction is ended with the mixed formulation of the domain decomposition of the problem. The problem presented is a model problem in order to keep the technicalities as much simple as possible. Section 2 is dedicated to the two asymptotic decompositions proposed for a given location of the interface Γ. One asymptotic decomposition is based on a particular mortar subspace (the constant functions on Γ), and the other one is based on coupling a partial differential equation with an ordinary differential equation. In section 3, a posteriori error estimates are given and an indicator is proposed. In section 4 the optimal location of the junction is found by minimizing the indicator. Numerical results are provided showing the efficiency of the proposed method.
Let f be a regular function defined by

$$f(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & 0 < x_1 < a \\ f_2(x_1) & a < x < 1 \end{cases} \tag{1.1}$$

and $g \in C^2(IR)$ an L-lipschitzian function. The domain $\Omega = (0,1) \times (0,1)$ is decomposed in two subdomains $\Omega_1 = (0,a) \times (0,1)$ and $\Omega_2 = (a,1) \times (0,1)$, the boundary $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, and the boundary $\partial\Omega$ is divided into four subparts $\gamma_1 = \{0\} \times (0,1)$ $\gamma_2 = (0,1) \times \{0\}$ $\gamma_3 = \{1\} \times (0,1)$ $\gamma_4 = (0,1) \times \{1\}$.



We consider the following semi linear problem: find $U \in H^2(\Omega)$ solution to:

$$\begin{cases} -\Delta U(x_1,x_2) + g(U) = f(x_1,x_2), \text{ in } \Omega \\ \partial_n U = 0 \text{ on } \gamma_{2i};\ 1 \le i \le 2; \\ U = 0 \text{ on } \gamma_{2i-1};\ 1 \le i \le 2; \end{cases} \tag{1.2}$$

Giving an exhaustive description of the existence results for semilinear elliptic problems is out of the scope of this work, see for example Lions (1982). So we will only consider the case $g : z \mapsto z^3$. It is straightforward to prove the existence and uniqueness of solutions to Problem (1.2) by minimizing over a closed subset of $H^1(\Omega)$, the convex lower semi continuous functional (see Ekland, Temam (1999) for example)

$$\int_\Omega |\nabla V|^2 + \frac{1}{4}V^4\,dx - \int_\Omega fV\,dx.$$

Now let us give a formulation of the problem (1.2) in the domain decomposition context with a L^2-mortar subspace. We define the following functional spaces where D_2 denotes the directional derivative with respect to the second variable.

$$\begin{array}{l} {}_0H^1(\Omega_1) = \{\varphi \in H^1(\Omega_1);\, \varphi|_{\gamma_1} = 0\};\\ {}_0H^1(\Omega_2) = \{\varphi \in H^1(\Omega_2);\, \varphi|_{\gamma_3} = 0\};\\ V ={}_0 H^1(\Omega_1) \times_0 H^1(\Omega_2)\\ W ={}_0 H^1(\Omega_1) \times_0 H^1(\Omega_2) \cap \{D_2\varphi|_{\Omega_2} = 0\};\\ \Lambda = L^2(\Gamma). \end{array} \tag{1.3}$$

equipped with the norms

$$|v|_1^2 = \sum_{i=1}^{2}\int_{\Omega_i} \nabla v_i \nabla v_i\,dx_1dx_2; \quad \|\xi\|_\Lambda^2 = \int_\Gamma \xi^2\,dx_2 \tag{1.4}$$

A L^2 setting is introduced for the Lagrange multiplier because the inner product of L^2 is easier to handle than the one of $H^{1/2}$. Moreover, numerically it is difficult to deal with the $H^{1/2}$ duality. Let us define $(u_1, u_2, \lambda) \in V \times \Lambda$ solution to

$$\left\{ \begin{array}{l} \sum_{i=1}^2 \int_{\Omega_i} \nabla u_i \nabla v_i + u_i^3 v_i\,dx_1dx_2 + \int_\Gamma \lambda(v_1 - v_2)\,dx_2 = \sum_{i=1}^2 \int_{\Omega_i} fv_i\,dx_1dx_2; \quad \forall v \in V\\ \int_\Gamma \xi(u_1 - u_2)\,dx_2 = 0 \quad \forall \xi \in \Lambda \end{array} \right. \tag{1.5}$$

We have the following result.

Lemma 1.1. *Assume $f \in L^2(\Omega)$ then, there exists a unique $(u_1, u_2, \lambda) \in V \times \Lambda$ solution to Problem (1.5). Moreover, we have $u_i = U|_{\Omega_i}$ for $1 \le i \le 2$.*

Proof. The proof proceed in two steps. First, the existence of $U \in K$ solution to Problem (1.7) is proved. Then the existence of the multiplier is investigated. Let us introduce the following trace operator:

$$\begin{array}{ll} B : V & \to \Lambda\\ (v_1, v_2) & \mapsto v_1 - v_2\Big|_\Gamma \end{array} \tag{1.6}$$

Let us denote by $\Lambda^\perp$ the orthogonal subspace to Λ according to the inner product of $L^2(\Gamma)$, and define K the closed subset of space V by

$$K = \{v \in V;\, v_1 - v_2\Big|_\Gamma \in \Lambda^\perp\}$$

Consider the following minimization problem

$$U = \mathrm{Argmin}_{V\in K} \sum_{i=1}^{2} \int_{\Omega_i} |\nabla V_i|^2 + \frac{1}{4}V_i^4\,dx - \int \Omega_i f V_i\,dx. \tag{1.7}$$

The functional to be minimized in (1.7) is convex lower semi continuous, the space K is convex, thus we have existence and uniqueness of a minimizer $U \in K$. Introduce the following bilinear forms:

$$\begin{array}{ll} a_u : V \times V \longrightarrow \mathbb{R} & \\ v, w & \mapsto a_u(v, w) = \sum_{i=1}^2 \int_{\Omega_i} \nabla v_i \nabla w_i + u_i^2 v_i w_i\,dx_1dx_2\\ b : \Lambda \times V & \longrightarrow \mathbb{R}\\ \xi, v & \mapsto b(\xi, v) = \int_\Gamma \xi(v_1 - v_2)\,dx_2. \end{array} \tag{1.8}$$

Observe that the bilinear positive definite form $a_u(\cdot,\cdot)$ define an inner product on V, the norm of which is equivalent to the H^1-semi-norm. Observe that Problem (1.5) expresses as

$$a_u(u,v) + b(\lambda, v) + b(\xi, u) = \sum_{i=1}^{2} \int_{\Omega_i} f v_i \, dx_1 dx_2; \quad \forall v \in V, \, \forall v \in \Lambda. \tag{1.9}$$

According to the inner product induced by $a_u(\cdot,\cdot)$, we have the following splitting for the space V:

$$K \bigoplus K^{\perp} = V, \tag{1.10}$$

with

$$K^{\perp} = \{v \in V, \, a_u(v,w) = 0, \, \forall w \in K\}.$$

Now we have to prove the existence of the Lagrange multiplier λ. Let us prove the following inf-sup condition: there exists $0 < \beta$ such that

$$\inf_{(w,\mu)\in K\times\Lambda} \sup_{(v,\xi)\neq(0,0)\in V\times\Lambda} \frac{a_u(w,v)+b(\mu,v)+b(\xi,w)}{\|\xi\|_\Lambda + |v|_1} \geq \beta. \tag{1.11}$$

Let (w,μ) be such that $|w|_1 + \|\mu\|_\Lambda = 1$.

- For $w \in K$, choose $v = w + w_1$ with w_1 solution to the problem:

$$\begin{cases} -\Delta w_1(x_1,x_2) = 0 \text{ in } \Omega_1 \\ \partial_n w_1 = \mu \text{ on } \Gamma \text{ and } w_1 = 0 \text{ on } \partial\Omega_1 \setminus \Gamma. \end{cases} \tag{1.12}$$

we have $(w_1, 0) \in K^{\perp}$ since assuming $(w_1, 0) \in K$ lead to $w_1|_\Gamma = 0$ which combined with the first equation of (1.12) would lead to $w_1 = 0$.
For all $v_1 \in H^1(\Omega_1), v_1 \mid_{\partial\Omega_1\setminus\Gamma} = 0$

$$\int_{\Omega_1} \nabla w_1 \nabla v_1 dx = \int_\Gamma \mu v_1 dx_2$$

and the following estimate holds true:

$$|w_1|_1^2 = \underset{\substack{v_1\in H^1(\Omega_1)\\ v_1|_{\partial\Omega_1\setminus\Gamma}=0}}{sup} \int_{\Omega_1} \nabla w_1 \nabla v_1 dx = \underset{\substack{v_1\in H^1(\Omega_1)\\ v_1|_{\partial\Omega_1\setminus\Gamma}=0}}{sup} \int_\Gamma \mu v_1 dx_2.$$

$$|w_1|_1 \leq c_2 \|\mu\|_{0,\Gamma}. \tag{1.13}$$

Since the space $\tilde{H}^{\frac{1}{2}}(\Gamma) = \{\varphi \in H^{\frac{1}{2}}(\Gamma)$ the extension of which by 0 belongs to $H^{\frac{1}{2}}(\partial\Omega_1)\}$ is densely embedded in $L^2(\Gamma)$, there exists $\mu_\varepsilon \in \tilde{H}^{\frac{1}{2}}(\Gamma)$: verifying:

$$\|\mu - \mu_\varepsilon\|_{0,\Gamma} \leq \varepsilon.$$

We choose ε such that :

$$|w_1|_1^2 = \underset{\substack{v_1\in H^1(\Omega_1)\\ v_1|_{\partial\Omega_1\setminus\Gamma}=0}}{sup} \int_\Gamma \mu_\varepsilon v_1 + v_1(\mu - \mu_\varepsilon) dx_2 \geq (\|\mu\|_\Gamma - \varepsilon)^2 - (\|\mu\|_{0,\Gamma} + \varepsilon)\varepsilon \geq \frac{1}{2}\|\mu\|_\Gamma^2.$$

So the following quantity

$$I = \sup_{(v,\xi)\neq(0,0)\in V\times\Lambda} \frac{a_u(w,v)+b(\mu,v)+b(\xi,w)}{\|\xi\|_\Lambda + |v|_1}$$

with $\xi = \mu$, $v = w + w_1$, and estimate (1.13) verifies the following estimation

$$I \geq \frac{|w|_1^2 + b(\mu,w_1)}{\|\mu\|_\Lambda + \|v\|_1} \geq \frac{|w|_1^2 + |w_1|_1^2}{\|\mu\|_\Lambda + \|v\|_1} \geq \frac{|w|_1^2 + \|\mu\|_{0,\Lambda}^2}{2(\|\mu\|_\Lambda + \|v\|_1)} \geq \frac{|w|_1^2 + \|\mu\|_\Lambda^2}{2(1+c_2)} \geq \frac{1}{4(1+c_2)} = \beta \tag{1.14}$$

Now let us establish that $(u_i - U)|_{\Omega_i} = 0$. Choose $v_i \in \mathcal{D}(\Omega_i)$, in Problem (1.5), we integrate by parts in the bilinear form $a_u(\cdot,\cdot)$. We deduce $\nabla(u_i - U)|_{\Omega_i} = 0$. Thanks to the Dirichlet's conditions on a part of the boundary, we have: $u_i - U|_{\Omega_i} = 0$.
Choose $v_i \in \mathcal{D}(\overline{\Omega}_i)$; $v_i|_{\gamma_{2i}} = 0$; $1 \leq i \leq 2$. Integrate by parts in the bilinear form $a_u(\cdot,\cdot)$ again, we have:

$$\int_\Gamma \partial_{n_1} u_1 v_1 + \partial_{n_2} u_2 v_2 + \lambda(v_1 - v_2)\, dx_2 = 0 \tag{1.15}$$

Take $v_1 = 0$ we have $\partial_{n_2} u_2 = \lambda$ in $L^2(\Gamma)$. Take $v_2 = 0$ we have $\partial_{n_1} u_1 = -\lambda$ in $L^2(\Gamma)$. The conditions are expressed in L^2 since, $u_i \in H^2(\Omega_i)$, $1 \leq i \leq 2$ due to the regularity of the Laplacian and the continuity of the injection $H^1 \hookrightarrow L^P$ for $1 < p < +\infty$.
Since $b(\xi, u) = 0$ for all $\xi \in L^2(\Gamma)$, we deduce that $u_1 = u_2$ on Γ.

□

Remark 1.1. Observe that the inf-sup conditions still holds true for a function g which satisfies: $g(0) = 0$, and $Dg(u)$ is semi definite positive.

2 Asymptotic Domain Decomposition

In this section, we propose two approximated domain decomposition problems by using different mortar subspaces or different spaces for the solution. Let $\Lambda_0 = \mathrm{span}\{1\}$ and let us define $(\tilde{u}_1, \tilde{u}_2, \lambda_0) \in V \times \Lambda_0$ solution to

$$\left\{ \begin{array}{l} a_0(\tilde{u}, v) + \sum_{i=1}^2 \int_{\Omega_i} \tilde{u}_i^3 v_i dx_1 dx_2 + b(\lambda_0, v) = \sum_{i=1}^2 \int_{\Omega_i} f v_i \, dx_1 dx_2; \quad \forall v \in V \\ b(\xi, \tilde{u}) = 0 \quad \forall \xi \in \Lambda_0. \end{array} \right. \tag{2.1}$$

Lemma 2.1. *Assume $f \in L^2(\Omega)$ then, there exists a unique $(\tilde{u}_1, \tilde{u}_2, \lambda_0) \in V \times \Lambda_0$ solution to Problem (2.1). Moreover, we have*

$$\partial_{n_1} \tilde{u}_1 = -\partial_{n_2} \tilde{u}_2 \textit{ in } L^2(\Gamma); \quad \tilde{u}_2|_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma \tilde{u}_1 \, dx_2.$$

.

Proof. The existence result is a consequence of the inf-sup condition, which is proved in the same way as in lemma 1 with the inner product $a_{\tilde{u}}(\cdot,\cdot)$ and $w_1 = cx_1$, and

$$\Lambda_0^\perp = \{\varphi \in L^2(\Gamma); \int_\Gamma \varphi(x_2) dx_2 = 0\} \Rightarrow (w_1, 0) \in K^\perp.$$

Integrate by parts in (2.1), thus since $\tilde{u}_i \in H^2(\Omega_i)$ we have

$$\partial_{n_2} \tilde{u}_2|_\Gamma = \lambda_0 \in \Lambda_0. \tag{2.2}$$

Take $v_2 = 0$, whatever v_1 is:

$$\int_\Gamma (\partial_{n_1} \tilde{u}_1 + \lambda_0) v_1 dx_2 = 0 \Rightarrow \partial_{n_1} \tilde{u}_1 + \lambda_0 = 0 \text{ in } \tilde{H}^{\frac{1}{2}}(\Gamma)' \Rightarrow \partial_{n_1} \tilde{u}_1 = -\lambda_0. \tag{2.3}$$

Now, let us prove that $\tilde{u} \in W$. Since λ_0 is constant, it is easy to prove that $\tilde{u}_2(x_1)$ solution to

$$\left\{ \begin{array}{l} -\tilde{u}_2''(x_1) + \tilde{u}_2^3(x_1) = f_2(x_1) \text{ in } a < x_1 < 1 \\ \tilde{u}_2'(a) = \lambda_0; \quad \tilde{u}_2(1) = 0; \end{array} \right. \tag{2.4}$$

is the unique solution $\tilde{u}_2$ in the domain Ω_2.
The condition $b(1, \tilde{u}) = 0$ implies $\tilde{u}_2(a) = \frac{1}{|\Gamma|} \int_\Gamma \tilde{u}_1 \, dx_2$.

□

Now, set $\Lambda_2 = L^2(\Gamma)$ as mortar subspace, and let us define $(\hat{u}_1, \hat{u}_2, \lambda_2) \in W \times \Lambda_2$ solution to

$$\begin{cases} a_0(\hat{u}, v) \sum_{i=1}^{2} \int_{\Omega_i} \hat{u}_i^3 v_i dx_1 dx_2 + b(\lambda_2, v) = \sum_{i=1}^{2} \int_{\Omega_i} f v_i \, dx_1 dx_2; \quad \forall v \in V \\ b(\xi, \hat{u}) = 0 \quad \forall \xi \in \Lambda_2 \end{cases} \tag{2.5}$$

Lemma 2.2. *Assume $f \in L^2(\Omega)$ then, there exists a unique $(\hat{u}_1, \hat{u}_2, \lambda_2) \in W \times \Lambda_2$ solution to Problem (2.5). Moreover, we have*

$$\partial_{n_2} \hat{u}_2 = -\frac{1}{|\Gamma|} \int_{\Gamma} \partial_{n_1} \hat{u}_1 \, dx_2 \quad \hat{u}_1 = \hat{u}_2 \text{ in } L^2(\Gamma).$$

Proof. The space W is a closed subspace of V thus the existence is proved in the same way as in Lemma 1.1. The identity equivalent to (1.15) identity with $v_1 = 0$ reads: for every $v_2 \in L^2(\Gamma)$

$$\int_{\Gamma} (\partial_{n_2} \hat{u}_2 - \lambda_2) v_2 dx_2 = 0. \tag{2.6}$$

Since $\partial_{n_2} \hat{u}_2 - \lambda_2 \in \Lambda_2^{\perp}$ we conclude that $\partial_{n_2} \hat{u}_2 = \lambda_2$. Take $v_2 = 0$, for every $v_1 \in L^2(\Gamma)$ The identity equivalent to (1.15) identity reads:

$$\int_{\Gamma} (\partial_{n_1} \hat{u}_1 + \lambda_2) v_1 dx_2 = 0 \Rightarrow \partial_{n_1} \hat{u}_1 = -\lambda_2 \text{ in } \Lambda_2. \tag{2.7}$$

Since $\hat{u}_2 \in W$, the relation (2.6) reads: $\partial_{n_2} \hat{u}_2 = -\frac{1}{|\Gamma|} \int_{\Gamma} \partial_{n_1} \hat{u}_1 \, dx_2$. The condition $b(\xi, \hat{u}) = 0$ for every $\xi \in \Lambda_2$ implies $\hat{u}_2 = \hat{u}_1$. □

3 A Posteriori Error Estimates

In this section an a posteriori error estimate is derived for the error between the exact solution of the domain decomposition formulation of the problem, and the approximated solution by using a mortar subspace.
Define the bilinear form $a_{u-\tilde{u}}(\cdot, \cdot)$ on $V \times V$ by:

$$a_{u-\tilde{u}}(w, v) = \sum_{i=1}^{2} \int_{\Omega_i} \nabla v_i \nabla w_i + \int_0^1 3(s u_i + (1-s)\tilde{u}_i)^2 \, ds w_i v_i \, dx_1 dx_2. \tag{3.1}$$

Introduce the error e:

$$e = (u - \tilde{u}, \lambda - \lambda_0). \tag{3.2}$$

For all $v \in V$, the error equation reads:

$$a_{u-\tilde{u}}((u - \tilde{u}), v) + b(\lambda - \lambda_0, v) = 0. \tag{3.3}$$

From (1.15) and the relation on Γ, $u_1 = u_2$, we have

$$-\int_{\Gamma} (\partial_{n_1} \tilde{u}_1 v_1 + \partial_{n_2} \tilde{u}_2 v_2 dx_2 - \int_{\Gamma} \lambda_0 (v_1 - v_2) dx_2 = 0. \tag{3.4}$$

Introduce the following linear form

$$\mathcal{L}_{u-\tilde{u},\lambda_0}(v, \xi) = -\int_{\Gamma} \xi (\tilde{u}_1 - \tilde{u}_2) \, dx_2 - \int_{\Gamma} (\partial_{n_1} \tilde{u}_1 v_1 + \partial_{n_2} \tilde{u}_2 v_2 dx_2 - \int_{\Gamma} \lambda_0 (v_1 - v_2) dx_2. \tag{3.5}$$

We have:

$$\mathcal{L}_{u-\tilde{u},\lambda_0}(v, \xi) = a_{u-\tilde{u}}((u - \tilde{u}), v) + b(\lambda - \lambda_0, v) + b(\xi, u - \tilde{u})$$

Lemma 3.1. *Assume $f \in L^2(\Omega)$ then, there exists $0 < C(u - \tilde{u}, \Omega)$ such that the following estimates hold true.*

$$C(u - \tilde{u}, \Omega)\|\mathcal{L}_{\tilde{u},\lambda_0}\|_* \leq \|e\| \leq \frac{\|\mathcal{L}_{\tilde{u},\lambda_0}\|_*}{\beta}, \tag{3.6}$$

with

$$\|\mathcal{L}_{\tilde{u},\lambda_0}\|_* = \|\tilde{u}_1 - \tfrac{1}{|\Gamma|}\textstyle\int_\Gamma \tilde{u}_1\, dx_2\|_{0,\Gamma} \tag{3.7}$$

and β is defined in (1.14).

Proof. We have to evaluate:

$$\|\mathcal{L}_{\tilde{u},\lambda_0}\|_* = \sup_{\substack{(v,\xi)\in V\times\Lambda \\ (v,\xi)\neq(0,0)}} \frac{a_{u-\tilde{u}}(u-\tilde{u},v)+b(\lambda-\lambda_0,v)+b(\xi,u-\tilde{u})}{\|\xi\|_{0,\Gamma}+|v|_1} \tag{3.8}$$

Gathering (3.8) with the previous Inf-Sup condition (1.14), we deduce

$$\|e\| \leq \frac{1}{\beta}\sup_{\substack{(v,\xi)\in V\times\Lambda \\ (v,\xi)\neq(0,0)}} \frac{-\int_\Gamma \partial_{n_1}\tilde{u}_1 v_1+\partial_{n_2}\tilde{u}_2 v_2 dx_2+b(-\lambda_0,v)+b(\xi,u-\tilde{u})}{\|\xi\|_{0,\Gamma}+|v|_1}. \tag{3.9}$$

which proves the bound from above in (3.6). Accounting for the relation between $\partial_{n_1}\tilde{u}_1$,$\partial_{n_1}\tilde{u}_2$ and λ_0 given in Lemma 2.1 we have:

$$\|e\| \leq \frac{1}{\beta}\sup_{\substack{(v,\xi)\in V\times\Lambda \\ (v,\xi)\neq(0,0)}} \frac{-\int_\Gamma \xi(\tilde{u}_1-\tilde{u}_2)\, dx_2}{\|\xi\|_{0,\Gamma}+|v|_1}, \tag{3.10}$$

and finally

$$\|e\| \leq \frac{1}{\beta}\|\tilde{u}_1 - \tilde{u}_2\|_{0,\Gamma} = \frac{1}{\beta}\|\tilde{u}_1 - \frac{1}{|\Gamma|}\int_\Gamma \tilde{u}_1\, dx_2\|_{0,\Gamma}. \tag{3.11}$$

The estimate from below is a consequence of the continuity of bilinear forms $a_{u-\tilde{u}}(\cdot,\cdot)$ and $b(\cdot,\cdot)$. □

The case with Λ_0 as mortar subspace is completed. Now, let us consider the second case where $f \in L^2(\Omega)$ and the mortar subspace is $\Lambda_2 = L^2(\Gamma)$. Define the bilinear forms $a_{u-\hat{u}}(\cdot,\cdot)$ on $W \times W$ as in (3.1), accounting for results in Lemma 2.2, and arguing in the same way as before we get the following indicator:

$$\|e\| \leq \frac{1}{\beta}\|\partial_{n_1}\hat{u}_1 + \partial_{n_2}\hat{u}_2\|_{0,\Gamma} = \frac{1}{\beta}\|\partial_{n_1}\hat{u}_1 - \frac{1}{|\Gamma|}\int_\Gamma \partial_{n_1}\hat{u}_1 dx_2\|_{0,\Gamma} \tag{3.12}$$

Remark 3.1. The estimate from below in (3.6) has no practical uses since it involves a constant which depends on the solution u.

Remark 3.2. The result of Lemma 3.1 strongly relies on the inf-sup condition which does not depends on the shape of the interface Γ.

4 Optimization with Respect to the Location of the Interface

Let a denote the position of the boundary Γ. Due to relation (3.11), the proposed strategy is to minimize with respect to a the functional $J(a)$ defined by:

$$J(a) = \|\tilde{u}_1(a, x_2) - \frac{1}{|\Gamma_a|}\int_{\Gamma_a} \tilde{u}_1(a, x_2)\, dx_2\|^2_{0,\Gamma}.$$

The algorithm 4 of minimization we propose is a simple descent algorithm. let a_0 and Tol be fixed.

- evaluate the derivative $DJ(a_n)$

- if $|DJ(a_n)| \leq Tol$ stop and if not
- $a_{n+1} = a_n - \theta.DJ(a_n)$ where θ is fixed number between 0 and 1.
- $n = n + 1$ return to the beginning.

Now we evaluate numerically the derivative with respect to the location of the boundary Γ. To compute $DJ(a_n)$ define:

$$I(a, x_2) = \tilde{u}_1(a, x_2) - \int_0^1 \tilde{u}_1(a, x_2)dx_2$$

Observe that

$$J(a) = (I(a, x_2), I(a, x_2))_{L^2(\Gamma)}$$

its derivative $DJ(a) = 2(\frac{\partial I(a,x_2)}{\partial a}, I(a, x_2)_{L^2(\Gamma)}$ where:

$$\frac{\partial I}{\partial a}(a, x_2) = \frac{\partial \tilde{u}_1}{\partial a}(a, x_2) - \int_0^1 \frac{\partial \tilde{u}_1}{\partial a}(a, x_2)dx_2.$$

To compute the derivative of $\tilde{u}_1$ with respect to $0 < a < 1$, the location of Γ_a we use the following change of geometry which consists in mapping the domain Ω with a moving boundary Γ_a onto a domain with a fixed boundary $\Gamma_{1/2}$. Thus the change of geometry will yield a change in coefficients of partial differential equations. Define the transformation T by

$$\begin{array}{lr} [0,1] \times [0,1] & \to [0,1] \times [0,1] \\ (z, x_2) & \to (x_1, x_2) = (T(z, a) = (2 - 4a)z^2 + (4a - 1)z, x_2) \end{array} \quad (4.1)$$

thus the segment $\Gamma_{\frac{1}{2}}$ is mapped to Γ_a. The unknown ψ is defined by $\psi = UoT$, the composition of U solution to Problem 1.2 with the change of variables T. The equation (1.2) becomes for function ψ:

$$\left\{ \begin{array}{l} -D_zT.D^2_{zz}\psi - (D_zT)^3.D^2_{x_2x_2}\psi + D^2_{zz}T.D_z\psi = (D_zT)^3 \left(f(T, x_2) - \psi^3\right) \\ \partial_n\psi = 0 \text{ on } \gamma_{2i};\ 1 \leq i \leq 2; \\ \psi = 0 \text{ on } \gamma_{2i-1};\ 1 \leq i \leq 2; \end{array} \right. \quad (4.2)$$

A variational formulation for the decomposed domain problem corresponding to the problem (4.2) with a mortar subspace Λ_0 is: $\Omega_1 = (0, \frac{1}{2}) \times (0, 1)$ and $\Omega_2 = (\frac{1}{2}, 1) \times (0, 1)$;

$$\left\{ \begin{array}{l} \sum_{i=1}^2 \int_{\Omega_i} c(z).\nabla\tilde{\Psi}_i.\nabla v_i\, dx_1dx_2 + 2\sum_{i=1}^2 \int_{\Omega_i} D^2_{zz}T.D_z\tilde{\Psi}_i.v_i\, dx_1dx_2 \\ +\int_{\Gamma_a} \lambda(v_1 - v_2)\, dx_2 = \sum_{i=1}^2 \int_{\Omega_i} (D_zT)^3 \left(f_i(T, x_2) - \tilde{\Psi}_i^3\right) v_i\, dx_1dx_2; \quad \forall v \in V \\ \int_\Gamma \xi(\tilde{\Psi}_1 - \tilde{\Psi}_2)\, dx_2 = 0 \quad \forall\xi \in \Lambda_0. \end{array} \right. \quad (4.3)$$

where c is diagonal 2×2 matrix such as $c_{11} = D_zT$ and $c_{22} = (D_zT)^3$. Now we calculate the derivative of the indicator $J(a)$ with respect to a with function $\tilde{\psi}$:

$$DJ(a) = 2\int_0^1 \left(\tilde{u}_1(a, x_2) - \int_0^1 \tilde{u}_1(a, x_2)\, dx_2\right) \frac{\partial I}{\partial a}(a, x_2)\, dx_2,$$

therefore:

$$DJ(a) = 2\int_0^1 [\tilde{\Psi}_1 - \int_0^1 \tilde{\Psi}_1 dx_2].[\tilde{\Psi}_{1a} - \int_0^1 \tilde{\Psi}_{1a} dx_2]dx_2,$$

where $\tilde{\Psi}_{ia}$ denotes the derivative with respect to a of function $\tilde{\Psi}_i$ for $1 \leq i \leq 2$.
In the case where $f \in L^2(\Omega)$ and $\Lambda_2 = L^2(\Gamma)$ we use the indicator given in (3.12), we have:

$$J(a) = \int_0^1 (\partial_{n_1}\hat{u}_1(a, x_2) - \int_0^1 \partial_{n_1}\hat{u}_1(a, x_2)dx_2)^2 dx_2 \quad (4.4)$$

$$DJ(a) = 2\int_0^1 [\partial_{x_1}\hat{\Psi}_1 - \int_0^1 \partial_{x_1}\hat{\Psi}_1 dx_2)][\partial_{x_1}\hat{\Psi}_{1a} - \int_0^1 \partial_{x_1}\hat{\Psi}_{1a} dx_2)]dx_2 \quad (4.5)$$

where $\hat{\Psi}_{ia}$ is the derive of $\hat{\Psi}_i$ with respect to the variable a.
Now take the derive of the equation (4.2) with respect to the variable a. We have:

$$-D_zT.D^3_{azz}\varPsi - (D_zT)^3.D^3_{ax_2x_2}\varPsi + D^2_{zz}T.D^2_{az}\varPsi = D^2_{az}T.D^2_{zz}\varPsi + 3D^2_{az}.(D_zT)^2.D^2_{x_2x_2}\varPsi$$
$$-D^3_{azz}T.D_z\varPsi + 3D^2_{az}T.(D_zT)^2\left(f(T,x_2) - \varPsi^3\right) + D_aT(D_zT)^3\left(D_{x_1}f(T,x_2) - 3D_a\varPsi\varPsi^2\right) \quad (4.6)$$

Define $\varPsi_a = D_a\Psi$ the equation (4.6) becomes:

$$\begin{cases} -D_zT.D_{zz}\Psi_a - (D_zT)^3.D^2_{x_2x_2}\Psi_a + D^2_{zz}T.D_z\Psi_a + 3D_aT(D_zT)^3\varPsi^2\Psi_a = D^2_{az}T.D^2_{zz}\Psi \\ \quad +3D^2_{az}T.(D_zT)^2.D^2_{x_2x_2}\Psi - D^3_{azz}T.D_z\Psi + 3D^2_{az}T.(D_zT)^2\left(f(T,x_2) - \varPsi^3\right) + \\ \quad D_aT(D_zT)^3D_{x_1}f(T,x_2) \\ \partial_n\Psi_a = 0 \text{ on } \gamma_{2i};\ 1 \le i \le 2; \\ \Psi_a = 0 \text{ on } \gamma_{2i-1};\ 1 \le i \le 2; \end{cases} \quad (4.7)$$

A variational formulation for Problem 4.7 in decomposed domain setting is:

$$\begin{cases} \sum_{i=1}^2 \int_{\Omega_i}(c(z)\nabla\tilde{\psi}_{ia}\nabla v_i + 2D^2_{zz}TD_z\tilde{\psi}_{ia}v_i) + 3D_aT(D_zT)^3\tilde{\psi}^2\tilde{\psi}_a v_i dx_1dx_2 + \int_{\Gamma_a}\lambda(v_1 - v_2)\,dx_2 \\ = \sum_{i=1}^2 \int_{\Omega_i}(-c_a(z)\nabla\tilde{\psi}_i\nabla v_i - 2D^3_{azz}TD_z\tilde{\psi}_iv_i)dx_1dx_2 + \\ \sum_{i=1}^2 \int_{\Omega_i}\left[3D^2_{az}T(D_zT)^2(f_i(T,x_2) - \tilde{\psi}_i^3) + D_aT(D_zT)^3D_{x_1}f_i(T,x_2)\right]v_idx_1dx_2 \quad \forall v \in V \\ \int_\Gamma \xi(\tilde{\psi}_{1a} - \tilde{\psi}_{2a})\,dx_2 = 0 \quad \forall\xi \in \Lambda_0 \end{cases} \quad (4.8)$$

where c is a 2×2 diagonal matrix such as $c_{11} = -D^2_{az}T$ and $c_{22} = -3D^2_{az}T(D_zT)^2$.
Now the section is ended with some numerical examples. Let define U by:

$$U(x_1,x_2) = \begin{cases} 10x_1[(x_1 - \frac{1}{2})^3.(x_2 - x_2^2)^2 + (1-x_1)^2] & 0 \le x_1 \le \frac{1}{2} \\ 10x_1(1-x_1)^2 & \frac{1}{2} \le x_1 \le 1 \end{cases} \quad (4.9)$$

and function f by:

$$f(x_1,x_2) = \begin{cases} 10[(12t-2)x_1^4 + (-18t+3)x_1^3 + (\frac{-3}{2} + 9t - 12t^2)x_1^2 + (9t^2 - \frac{3}{2}t - \frac{23}{4})x_1 \\ -\frac{3}{2}t^2 + 4] + 1000x_1^3[t^2(x_1 - \frac{1}{2})^3 + (1-x_1)]^3 & 0 < x_1 < \frac{1}{2} \\ 1000x_1^3(1-x_1)^6 + 10(-6x_1 + 4) & \frac{1}{2} < x_1 < 1 \end{cases} \quad (4.10)$$

with:$t = (x_2 - x_2^2)^2$. It is straightforward to check that U solves

$$\begin{cases} -\Delta U(x_1,x_2) + U(x_1,x_2)^3 = f(x_1,x_2), \text{ in } \Omega \\ \partial_n U = 0 \text{ on } \gamma_{2i};\ 1 \le i \le 2; \\ U = 0 \text{ on } \gamma_{2i-1};\ 1 \le i \le 2. \end{cases} \quad (4.11)$$

Observe that U solves a domain decomposition formulation with an interface Γ located at $a = \frac{1}{2}$.
Let us define V_h respectively W_h the spaces V respectively W approximated with a triangular Lagrange finite element method of order one, the maximal size of triangles diameter is $h = 10^{-1}$. The space Λ_{2_h} is constituted with the traces of space V_h on the interface Γ_a. Let $(u_h, \lambda_h)) \in V_h \times \Lambda_{2_h}$ be solution to

$$\begin{cases} \sum_{i=1}^2 \int_{\Omega_i}\nabla u_{i_h}\nabla v_{i_h} + u_{i_h}^3 v_{i_h}\,dx_1dx_2 + \int_\Gamma \lambda(v_{1_h} - v_{2_h})\,dx_2 = \sum_{i=1}^2 \int_{\Omega_i} f v_{i_h}\,dx_1dx_2; \quad \forall v_h \in V_h \\ \int_\Gamma \xi_h(u_{1_h} - u_{2_h})\,dx_2 = 0 \quad \forall\xi_h \in \Lambda_{2_h}. \end{cases} \quad (4.12)$$

For solving numerically Problem (4.12) a fixed point strategy is used. In Figure 1 the error between U and u_h is represented for the three first iterations of the fixed point, and for the converged solution. The discontinuity of the error through Γ is due to the discontinuity of the approached solution through

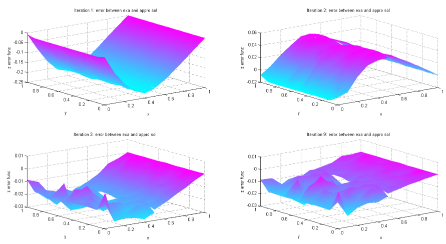


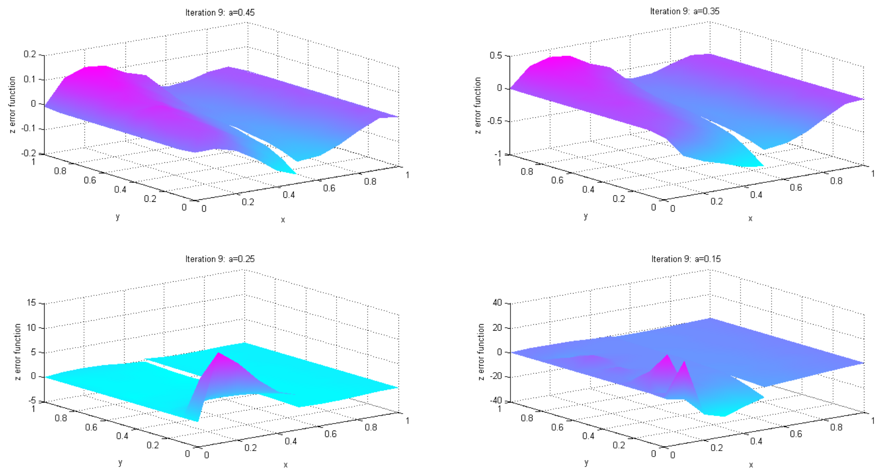
Figure 1: error function

Γ .

Now let us define $(\tilde{u}_h, \lambda_0)) \in V_h \times \{C^{\mathrm{st}}\}$ be solution to

$$\begin{cases} \sum_{i=1}^{2} \int_{\Omega_i} \nabla \tilde{u}_{i_h} \nabla v_{i_h} + \tilde{u}_{i_h}^3 v_{i_h}\, dx_1 dx_2 + \int_\Gamma \lambda_0 (v_{1_h} - v_{2_h})\, dx_2 = \sum_{i=1}^{2} \int_{\Omega_i} f v_{i_h}\, dx_1 dx_2; \quad \forall v_h \in V_h \\ \int_\Gamma \xi (\tilde{u}_{1_h} - \tilde{u}_{2_h})\, dx_2 = 0 \quad \forall \xi \in \{C^{\mathrm{st}}\}. \end{cases} \tag{4.13}$$

In Figure 4 the error between the exact solution and the solution to the domain decomposition problem (4.13), is presented for four locations of Γ_a.



Define the indicator by:

$$J(a) = \|\tilde{u}_{1_h}(a, x_2) - \frac{1}{|\Gamma_a|} \int_{\Gamma_a} \tilde{u}_{1_h}(a, x_2)\, dx_2\|_{0,\Gamma}^2.$$

Problem (4.8) is approximated in $V_h \times \{C^{\mathrm{st}}\}$, then $J(a)$ is computable whatever a is. In Figure 2, the curve $a \mapsto J(a)$ is presented.

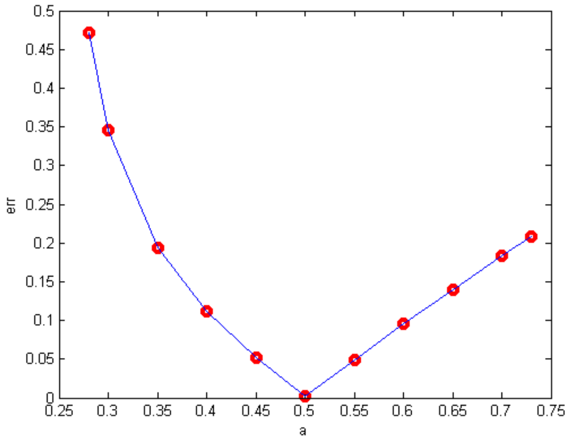


Figure 2: Indicator for mortar subspace Λ_0

The algorithm 4 defined in section **??** has been implemented, and the derivative $DJ(a_n)$ has been computed by solving Problem (4.8) approximated with a triangular Lagrange finite element method of order one. In Figure 3-4 convergence curves are presented for starting points $a_0 = .35$ and $a_0 = .65$ with a mortar subspace Λ_0.

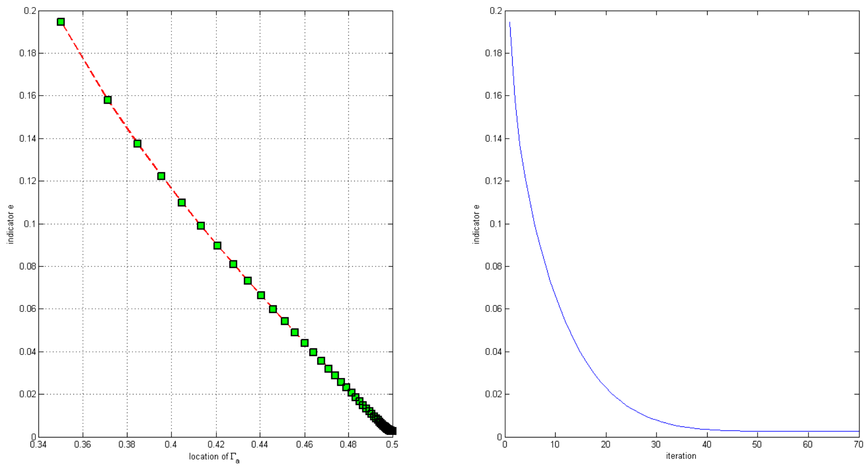


Figure 3: Indicator as function of location of the interface left and position of the interface as function of iterations right

Now let us come to the second mortar subspace presented. Let be $f \in L^2(\Omega)$ and $((\hat{u}_h, \lambda_2) \in$

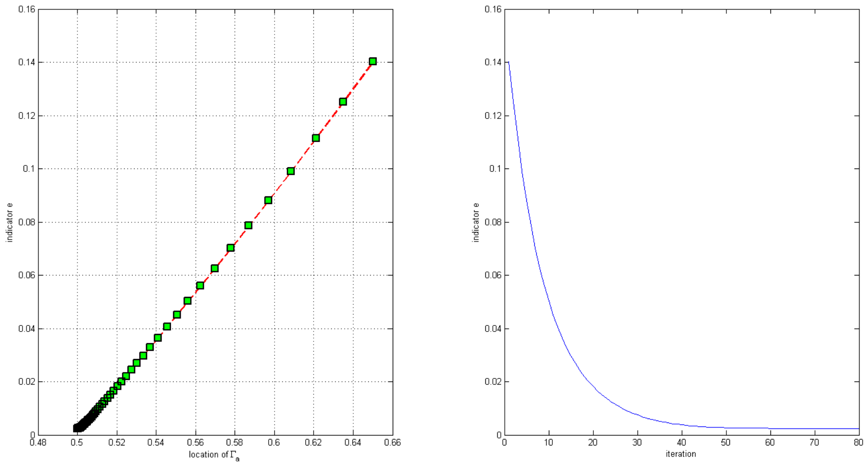


Figure 4: Indicator as function of location of the interface left and location as function of iterations right

$W_h \times \Lambda_{2_h}$ be solution to

$$\left\{ \begin{array}{l} \sum_{i=1}^{2} \int_{\Omega_i} \nabla \hat{u}_{i_h} \nabla v_{i_h} + \hat{u}_{i_h}^3 v_{i_h} \, dx_1 dx_2 + \int_\Gamma \lambda_0 (v_{1_h} - v_{2_h}) \, dx_2 = \sum_{i=1}^{2} \int_{\Omega_i} f v_{i_h} \, dx_1 dx_2; \quad \forall v_h \in W_h \\ \int_\Gamma \xi (\hat{u}_{1_h} - \hat{u}_{2_h}) \, dx_2 = 0 \quad \forall \xi \in \Lambda_{2_h} \}. \end{array} \right. \tag{4.14}$$

In Figure 5 the error between the exact solution and the approached solution for the three first iterations of the fixed point and for the converged solution are given. Define the indicator by:

$$J(a) = \int_0^1 (\partial_{n_1} \hat{u}_{1_h}(a, x_2) - \int_0^1 \partial_{n_1} \hat{u}_{1_h}(a, x_2) dx_2)^2 dx_2 \tag{4.15}$$

In Figure 6, the indicator is plotted as function of the location of the interface.

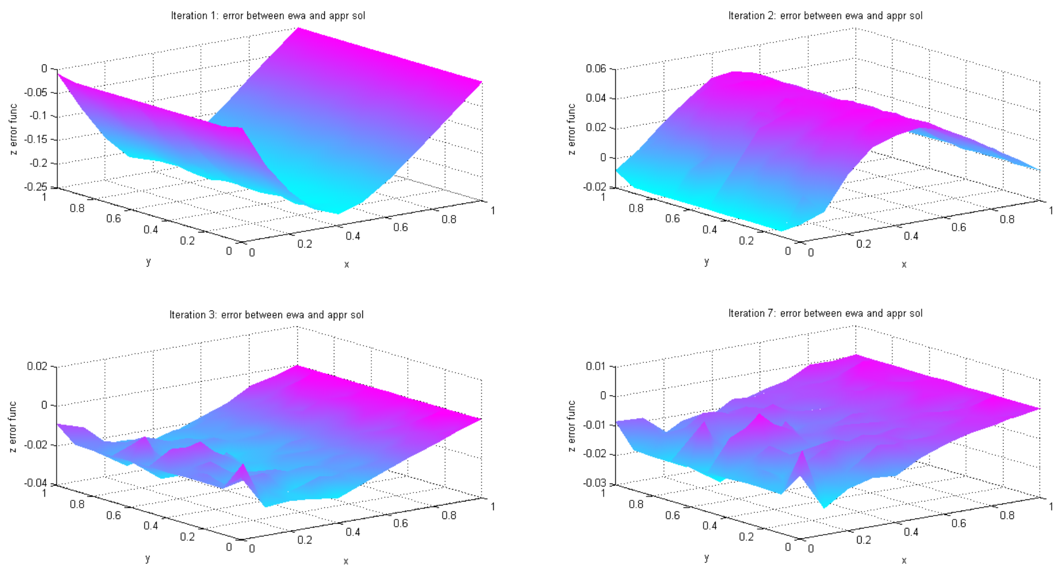


Figure 5: error with a mortar subspace Λ_2

Doing in the same way as before, in Figure 7 the indicator is represented as function of the position of the interface for a starting position $a_0 = .35$, and the location of the interface is described as function of the iterations.

Let us conclude this paper with some computational considerations.

- The proposed indicator is computed only with u_{h_1} the approximated solution in domain Ω_1.
- When dealing with a 2D or 3D domain linked with 1D domains, that is to say when considering a semi linear PDE linked with ODE's, for a prescribed accuracy the question is: does it better to change the locations of the interfaces of the domain Ω_1 or does it better to refine the mesh in the domain Ω_1. The answer is quite simple, by using your favorite indicator of the mesh error , you compare it to the indicators of the location error proposed in this article. Then you are able to decide if you should refine the mesh or if you should move the interfaces. For example, for Problem 4.13 an accuracy of 10^{-2} is required. Let us start with an interface located at $a = .35$ and with a size of mesh of 10^{-1}. Compute the indicator of the location error we get 0.1944, and the indicator of the mesh error is valued between 0.0203 and 0.0268 (see Table 8). Thus the interface is moved to $a = .4$ in order to enlarge the size of the domain Ω_1. The indicator of the location error becomes 0.1122, and the indicator of the mesh error is valued between 0.0512 and 0.0264. The mesh is then refined in the domain Ω_1 with a mesh size of 510^{-2} and the indicator of the mesh error is valued between 0.0147 and 0.0831. The interface is now moved to $a = .45$, the indicator of the location error becomes 0.0514. In Figure 10, the mesh of domain Ω_1 is presented. The mesh refinement strategy is quite crude, since the mesh is uniformly refined.
- Observe that whatever the values of the indicator of the mesh error is, it is possible to reach the optimal location of the interface.

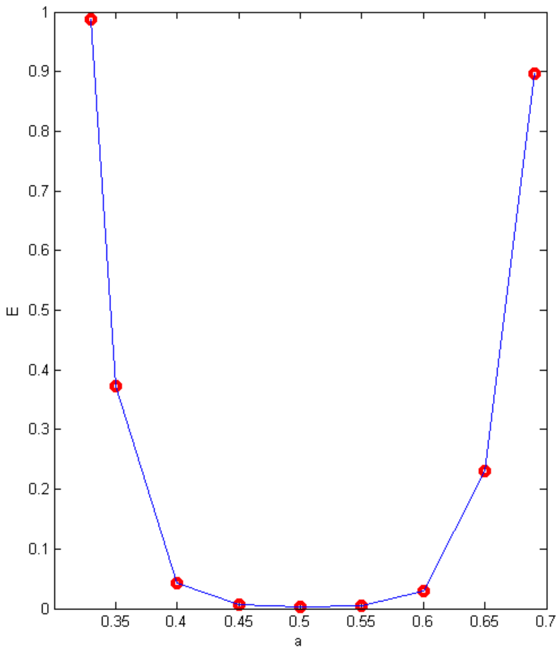


Figure 6: error evaluation in case Λ_2

The shape of the interface Γ is basically governed by the asymptotic properties of the solution U, depending on the geometry of the domain for example. In the presented problem, the interface is very simple, but well suited to the aim to couple a semi linear 1D problem with a 2D problem. If the interface were a curve, the presented results could be generalized at the expense of a more complicated derivative Problem 4.7, since the change of variables would induce a PDE problem with full variable coefficients. If from asymptotic analysis it is known that the solution U does not depend on the second variable for $x_1 > a$, then a curved interface completely immersed in $x_1 > a$ could be considered and for example the results of Section **??** will remain unchanged.

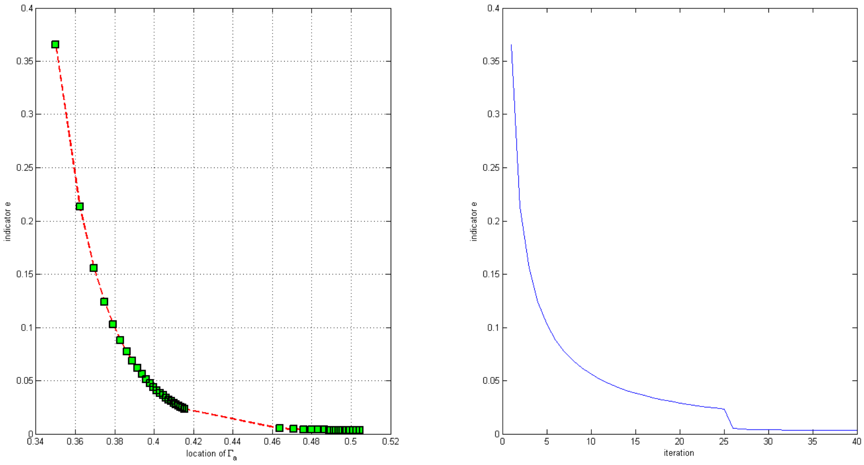


Figure 7: Indicator as function of location left and position of the interface as function of iterations right

h	a	location ind	min err indic mesh	max err indic mesh
10^{-1}	0.45	0.0518	0.0708	0.3153
	0.40	0.1122	0.0512	0.2646
	0.35	0.1944	0.0203	0.2687
5.10^{-2}	0.45	0.0514	0.0147	0.0831
	0.40	0.1117	0.0115	0.0705
	0.35	0.1949	0.0040	0.0730

Figure 8: Location indicator and mesh error indicator for two meshes

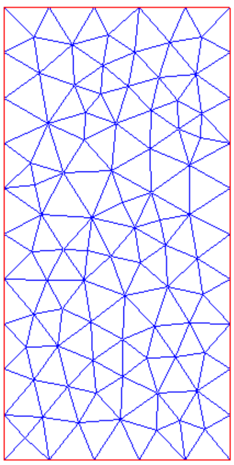


Figure 9: mesh of domain Ω_1

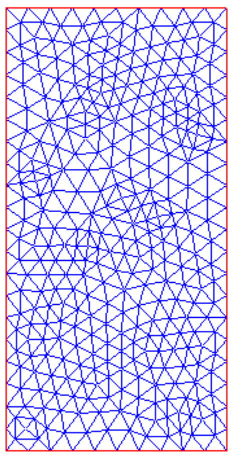


Figure 10: refine mesh of domain Ω_1

References

Becker, R., Rannacher, R. (2003). An optimal control approach to a posteriori error estimation in finite element methods. Acta Numerica Cambridge University Press, 1–102.

Blanco, P., Leiva, J., Feijóo, R., Buscaglia, G. (2010). Black-box decomposition approach for computational hemodynamics: One-dimensional models. Comp. Meth. Appl. Mech. and Engineering, 200, 1389–1405.

Buscaglia, G., Feijóo, R., Padra, C. (1995). A posteriori error estimation in sensitivity analysis. Structural and Multidisciplinary Optimization, 9, 194–199.

Ekland, I., Temam, R. (1999). Convex analysis and variational problems. SIAM Classics In applied mathematics.

Fontvieille, F. (2004). Décomposition Asymptotique et éléments finis. Phd thesis INSA de Lyon 04ISAL0029.

Fontvielle, F., Panassenko, G., Pousin, J. (2007). F.E.M. implementation for the asymptotic partial decomposition. Applicable Analysis, 86, 5, 519–536.

Guermont, J.L., Ern, A. (2000). Elément finis: théorie, applications, mise en oeuvre. Springer.

Leiva, J., Blanco, P., Buscaglia, G. (2009). Iterative strong coupling of dimensionally heterogeneous models. Int. J. Numer. Meth. Engineering, 81, 1558–1580.

Leiva, J., Blanco, P., Buscaglia, G. (2011). Partitioned analysis for dimensionally-heterogeneous hydraulic networks. SIAM Multiscale Modeling and Simulation, 9, 872–903.

Lions, P.L. (1982). On the existence of the positive solutions of semilinear elliptic equations. SIAM Riview, 24, 4 441–467.

Panasenko, G.P. 1996. Method of asymptotic partial decomposition of a domain. Math.Models Methods Appl. Sci., 1, 8, 139-156.

Panasenko, G.P. (1999). Asymptotic partial decomposition of variational problem. C. R. Acad. Sci. Paris February, 327, Série IIb, 1185–1190.

Panasenko, G.P. (2005). Multi-scale modelling for structures and composites. Springer, the Netherlands.

Quarteroni, A., Veneziani, A. (2011). Analysis of a geometrical multiscale model based on the coupling of PDE's and ODE's for Blood Flow Simulations. SIAM J. on MMS., 1, 2, 173–195.

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