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Certain Subclass of k-Uniformly p-Valent Starlike and Convex Functions Associated with Fractional Derivative Operators

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Research Article

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Abstract

In this paper, we introduce a new subclass $k-UCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$ of k-uniformly p-valent starlike and convex functions in the open unit disk using a fractional differential operator. We obtain coefficient estimates, distortion theorems, extermal properties, closure theorems, and inclusion properties. The radii for k-uniformly starlikeness, convexity and close-to-convexity for functions belonging to this class are also determined.

Keywords: p-valent function; starlike function; convex function; k-uniformly starlike function; k-uniformly convex function; fractional derivative operators

2010 Mathematics Subject Classification: 30C45; 26A33

1 Introduction

The classes of uniformly convex (starlike) functions were first introduced by [Goodman (1991a,b)] , and were studied subsequently by [Rønning (1991, 1993)], [Ma and Minda (1992, 1993)], and others. Also, the classes of k-uniformly convex (starlike) functions were studied by [Kanas and Wisniowska (1999, 2000)]; where their geometric definitions and connections with the conic domains were considered. More recently, [Murugusundaramoorthy and Themangani (2009)], presented a study of interesting class $UCV(\alpha,\beta,\gamma)$ of uniformly convex functions based on certain fractional derivative operator, motivated by the earlier works of [Altintas, et al. (1995a,b)], [Owa, et al. (1989)], and [Raina and Srivastava (1996)].

In the present paper, we define a new subclass $k-UCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$ of k-uniformly p-valent starlike and convex functions in the open unit disk by making use of certain fractional derivative operator. We establish several properties like coefficient estimates, distortion theorems, extremal properties, closure theorems, and inclusion properties. Finally we determine the radii of k-uniformly starlikeness, convexity and close-to-convexity for functions belonging to this class.

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2 Preliminaries and Definitions

Let $\mathcal{A}(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbf{N}) \tag{2.1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Also, denote by $T(p)$ the subclass of $\mathcal{A}(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geq 0,\ p \in \mathbf{N}) \tag{2.2}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be k-uniformly p-valent starlike of order $\alpha,\ (-p < \alpha < p),\ k \geq 0$ and $z \in \mathcal{U}$, denoted by $k - UST(p, \alpha)$, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq k\left|\frac{zf'(z)}{f(z)} - p\right| \tag{2.3}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be k-uniformly p-valent convex of order $\alpha,\ (-p < \alpha < p),\ k \geq 0$ and $z \in \mathcal{U}$, denoted by $k - UCV(p, \alpha)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq k\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \tag{2.4}$$

In particular, when p=1, we obtain $k - UST(\alpha)$ and $k - UCV(\alpha)$, the classes of k-uniformly starlike and k-uniformly convex functions of order $\alpha,\ (-1 < \alpha < 1)$, respectively which were studied by various authors including [Owa (1998)], [Murugusundaramoorthy and Themangani (2009)], [Kanas and Wisniowska (2000)], [Rønning (1991)] and [Khairnar and Meena More (2009)].

Let ${}_2F_1(a, b; c; z)$ be the Gauss hypergeometric function defined for $z \in \mathcal{U}$ by, [Srivastava and Karlsson (1985)]

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{2.5}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & \text{when } n = 0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & \text{when } n \in \mathbf{N}. \end{cases} \tag{2.6}$$

for $\lambda \neq 0, -1, -2, \ldots$

We recall the following definitions of fractional derivative operators which were used by [Owa (1978)], see also [Altintas, et al. (1995a,b)] and [Raina and Srivastava (1996)] as follows:

Definition 2.1. The fractional derivative of order λ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \tag{2.7}$$

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of $(z - \xi)^{-\lambda}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

Definition 2.2. Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbf{R}$. Then, in terms of the familiar Gausss hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi\right) \tag{2.8}$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^\varepsilon),\ z \to 0$, where $\varepsilon > \max\{0, \mu - \eta\} - 1$ and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^\lambda f(z), \quad 0 \le \lambda < 1 \tag{2.9}$$

Now we define a new subclass of k-uniformly p-valent starlike and convex functions based on fractional derivative operator.

Definition 2.3. The function $f(z) \in \mathcal{A}(p)$ is said to be in the class $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ if and only if

$$\mathrm{Re}\left\{ \frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - \alpha \right\} \ge k \left| \frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - p \right|, \quad z \in \mathcal{U} \tag{2.10}$$

for $k \ge 0\,;\ 0 \le \alpha < p\,;\ \lambda \ge 0\,;\ 0 \le \mu < 1 + p\,;\ \beta \ge 0\,;\ 0 \le \gamma < 1 + p\,;\ \eta > \max(\lambda, \mu) - p - 1\,;\ \xi > \max(\beta, \gamma) - p - 1$. Denoted by $M_{o,z}^{\lambda,\mu,\eta} f(z)$ and $M_{0,z}^{\beta,\gamma,\xi} f(z)$ the modifications of the fractional derivative operator for the function $f(z)$ which are defined in terms of $J_{o,z}^{\lambda,\mu,\eta}$ and $J_{o,z}^{\beta,\gamma,\xi}$, respectively,as follows:

$$M_{o,z}^{\lambda,\mu,\eta} f(z) = \phi_p(\lambda, \mu, \eta) z^\mu J_{o,z}^{\lambda,\mu,\eta} f(z) \tag{2.11}$$

with

$$\phi_p(\lambda, \mu, \eta) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)} \tag{2.12}$$

also, we let

$$k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha) = k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha) \cap T(p) \tag{2.13}$$

The above classes $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ and $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ are of special interest and they contain many well-known classes of analytic functions. In particular; For $\mu = \lambda,\ \gamma = \beta,\ k = 1$ and $p = 1$, we have

$$1 - UCV_{\beta,\beta,\xi}^{\lambda,\lambda,\eta}(1,\alpha) = UCV(\lambda,\beta,\alpha)$$

and

$$1 - TUCV_{\beta,\beta,\xi}^{\lambda,\lambda,\eta}(1,\alpha) = TUCV(\lambda,\beta,\alpha)$$

where $UCV(\lambda,\beta,\alpha)$ and $TUCV(\lambda,\beta,\alpha)$ are precisely the subclasses of uniformly convex functions which were studied by [Murugusundaramoorthy and Themangani (2009)].

Furthermore, by specifying the parameters $\lambda, \mu, \beta, \gamma, \alpha, k$ and p, we obtain the following subclasses which were studied by various other authors:

1. For $\mu = \lambda = 1,\ \beta = \gamma = 0$, and $k = 1$, the class $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ can be reduced to $UST(p,\alpha)$ the class of uniformly p-valent starlike functions of order α, [Al-Kharsani and AL-Hajiry (2006)].
2. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ p = 1$, and $k = 1$, we obtain $UST(\alpha)$ the class of uniformly starlike functions of order α, see [Owa (1998)] and [Rønning (1991)].
3. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ \alpha = 0,\ p = 1$, and $k = 1$, we obtain UST the class of uniformly starlike functions, [Goodman (1991b)].
4. For $\mu = \lambda = 1,\ \beta = \gamma = 0$, and $k = 0$, we obtain the class of all p-valent starlike functions of order $\alpha,\ S^*(p,\alpha)$, [Partil and Thakare (1983)].
5. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ p = 1$, and $k = 0$, we have the class of starlike functions of order $\alpha,\ S^*(\alpha)$, see [Duren (1983)], [Jack (1971)], [Robertson (1936)], [Pinchuk (1968)] and [Schild (1965)].
6. For $\mu = \lambda = 1,\ \beta = \gamma = 0,\ \alpha = 0,\ p = 1$ and $k = 0$, we have the class of starlike functions S^*, [Duren (1983)].

In order to prove our results we mention to the following known result which shall be used in the sequel [Raina and Srivastava (1996)]

Lemma 2.1. *Let* $\lambda, \mu, \eta \in \mathbf{R}$*, such that* $\lambda \geq 0$ *and* $K > \max\{0, \mu - \eta\} - 1$. *Then*

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu} \tag{2.14}$$

3 Coefficient Estimates

Theorem 3.1. *The function* $f(z)$ *defined by* (2.1) *is in the class* $k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p, \alpha)$ *if*

$$\sum_{n=1}^{\infty} [p(1+k)\delta_n(\lambda, \mu, \eta, p) - (pk+\alpha)\delta_n(\beta, \gamma, \xi, p)] \, |a_{p+n}| \leq p - \alpha \tag{3.1}$$

where

$$\delta_n(\lambda, \mu, \eta, p) = \frac{\phi_p(\lambda, \mu, \eta)}{\phi_{p+n}(\lambda, \mu, \eta)} = \frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n} \tag{3.2}$$

and

$$\delta_n(\beta, \gamma, \xi, p) = \frac{\phi_p(\beta, \gamma, \xi)}{\phi_{p+n}(\beta, \gamma, \xi)} = \frac{(1+p)_n(1+\xi-\gamma+p)_n}{(1-\gamma+p)_n(1+\xi-\beta+p)_n} \tag{3.3}$$

with $\phi_p(\lambda, \mu, \eta)$ *and* $\phi_p(\beta, \gamma, \xi)$ *are given by* (2.12).

Proof. Applying Lemma (2.1), we have from (2.1) and (2.11) that

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n}$$

and

$$M_{0,z}^{\beta,\gamma,\xi} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\beta, \gamma, \xi, p) a_{p+n} z^{p+n}$$

Since $f(z) \in k - UCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p, \alpha)$, it suffices to show that

$$k \left| \frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - p \right| - \mathrm{Re} \left\{ \frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - p \right\} \leq p - \alpha$$

Notice that

$$k \left| \frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - p \right| - \mathrm{Re} \left\{ \frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - p \right\} \leq (1+k) \left| \frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - p \right|$$

$$\leq (1+k) \left| \frac{\sum_{n=1}^{\infty} p[\delta_n(\lambda, \mu, \eta, p) - \delta_n(\beta, \gamma, \xi, p)] a_{p+n} z^{p+n}}{z^p + \sum_{n=1}^{\infty} \delta_n(\beta, \gamma, \xi, p) a_{p+n} z^{p+n}} \right|$$

$$\leq (1+k) \frac{\sum_{n=1}^{\infty} p[\delta_n(\lambda, \mu, \eta, p) - \delta_n(\beta, \gamma, \xi, p)] |a_{p+n}|}{1 - \sum_{n=1}^{\infty} \delta_n(\beta, \gamma, \xi, p) |a_{p+n}|}$$

The last inequality above is bounded by $(p - \alpha)$ if

$$\sum_{n=1}^{\infty} [p(1+k)\delta_n(\lambda, \mu, \eta, p) - (pk+\alpha)\delta_n(\beta, \gamma, \xi, p)] |a_{p+n}| \leq p - \alpha$$

This completes the proof. □

Next, we state and prove the necessary and sufficient condition for $f(z)$ to be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$.

Theorem 3.2. *The function $f(z)$ defined by (2.2) is in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)\right] a_{p+n} \leq p - \alpha \tag{3.4}$$

where $\delta_n(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p)$ are given by (3.2) and (3.3) respectively. The result (3.4) is sharp.

Proof. In view of Theorem 3.1, we need to prove the sufficient part. Let $f(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ and z be real, then by the inequality (2.10)

$$\mathrm{Re}\left\{\frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - \alpha\right\} \geq k \left|\frac{pM_{o,z}^{\lambda,\mu,\eta} f(z)}{M_{o,z}^{\beta,\gamma,\xi} f(z)} - p\right|, \quad z \in \mathcal{U}$$

$$\frac{p - \sum_{n=1}^{\infty} p\delta_n(\lambda,\mu,\eta,p)a_{p+n}z^n}{1 - \sum_{n=1}^{\infty} \delta_n(\beta,\gamma,\xi,p)a_{p+n}z^n} - \alpha \geq k \left|\frac{\sum_{n=1}^{\infty} p[\delta_n(\lambda,\mu,\eta,p) - \delta_n(\beta,\gamma,\xi,p)]a_{p+n}z^n}{1 - \sum_{n=1}^{\infty} \delta_n(\beta,\gamma,\xi,p)a_{p+n}z^n}\right|$$

Letting $z \to 1$ along the real axis, we obtain

$$\frac{(p-\alpha) - \sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]a_{p+n}}{1 - \sum_{n=1}^{\infty} \delta_n(\beta,\gamma,\xi,p)a_{p+n}} \geq 0$$

This is only possible if (3.4) holds. Therefore we obtain the desired result. The result (3.4) is sharp for

$$f(z) = z^p - \frac{p-\alpha}{p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)} z^{p+n} \qquad (p, n \in \mathbf{N}) \tag{3.5}$$

□

Corollary 3.3. *Let the function $f(z)$ defined by (2.2) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, then*

$$a_{p+n} \leq \frac{p-\alpha}{p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)} \qquad (p, n \in \mathbf{N}) \tag{3.6}$$

with equality for the function $f(z)$ given by (3.5).

4 Distortion Theorems

Theorem 4.1. *Let the function $f(z)$ defined by (2.2) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ such that $k \geq 0;\ 0 \leq \alpha < p;\ \lambda \geq 0;\ 0 \leq \mu < 1+p;\ \beta \geq 0;\ 0 \leq \gamma < 1+p;\ \gamma \leq \mu,\ \ \eta \geq \lambda\left(1 - \frac{2+p}{\mu}\right)$ and $\xi \geq \beta\left(1 - \frac{2+p}{\gamma}\right)$. Then*

$$|z|^p - A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)|z|^{p+1} \leq |f(z)| \leq |z|^p + A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)|z|^{p+1} \tag{4.1}$$

where

$$A_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha) = \frac{p-\alpha}{p(1+k)\delta_1(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_1(\beta,\gamma,\xi,p)} \tag{4.2}$$

The estimates for $|f(z)|$ are sharp.

Proof. We observe that the functions $\delta_n(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p)$ defined by (3.2) and (3.3), respectively, satisfy the inequalities $\delta_n(\lambda,\mu,\eta,p) \le \delta_{n+1}(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p) \le \delta_{n+1}(\beta,\gamma,\xi,p),\ \forall n \in \mathbf{N}$ provided that $\eta \ge \lambda\left(1-\frac{2+p}{\mu}\right)$ and $\xi \ge \beta\left(1-\frac{2+p}{\gamma}\right)$. So $\delta_n(\lambda,\mu,\eta,p)$ and $\delta_n(\beta,\gamma,\xi,p)$ are non-decreasing functions.

$$0 < \frac{(1+p)(1+\eta-\mu+p)}{(1-\mu+p)(1+\eta-\lambda+p)} = \delta_1(\lambda,\mu,\eta,p) \le \delta_n(\lambda,\mu,\eta,p), \quad \forall n \in \mathbf{N} \tag{4.3}$$

also

$$0 < \frac{(1+p)(1+\xi-\gamma+p)}{(1-\gamma+p)(1+\xi-\beta+p)} = \delta_1(\beta,\gamma,\xi,p) \le \delta_n(\beta,\gamma,\xi,p), \quad \forall n \in \mathbf{N} \tag{4.4}$$

Since $f(z) \in k-TUCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$ then

$$\begin{aligned}&[p(1+k)\delta_1(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_1(\beta,\gamma,\xi,p)]\sum_{n=1}^{\infty}a_{p+n} \le \\ &\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]a_{p+n} \le p-\alpha\end{aligned} \tag{4.5}$$

So that (4.5) reduces to

$$\begin{aligned}\sum_{n=1}^{\infty}a_{p+n} &\le \frac{p-\alpha}{p(1+k)\delta_1(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_1(\beta,\gamma,\xi,p)} \\ &= A^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)\end{aligned} \tag{4.6}$$

From (2.2), we obtain

$$|f(z)| \le |z|^p + |z|^{p+1}\sum_{n=1}^{\infty}a_{p+n} \tag{4.7}$$

and

$$|f(z)| \ge |z|^p - |z|^{p+1}\sum_{n=1}^{\infty}a_{p+n} \tag{4.8}$$

on using (4.6) to (4.7) and (4.8), we arrive at the desired results (4.1).

Finally, we can see that the estimate for $|f(z)|$ are sharp by taking the function

$$f(z) = z^p - \frac{p-\alpha}{p(1+k)\delta_1(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_1(\beta,\gamma,\xi,p)}z^{p+1} \tag{4.9}$$

This completes the proof of Theorem 4.1. □

Corollary 4.2. *Let the function $f(z)$ defined by (2.2) be in the class $k-TUCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < R_1$, where*

$$R_1 = 1 - \frac{p-\alpha}{p(1+k)\delta_1(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_1(\beta,\gamma,\xi,p)} \tag{4.10}$$

5 Extremal Properties

Theorem 5.1. *Let $f_p(z) = z^p$ and*

$$f_{p+n}(z) = z^p - \frac{p-\alpha}{p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)}z^{p+n}, \quad (p,\ n \in \mathbf{N}) \tag{5.1}$$

Then $f(z) \in k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}\theta_{p+n}f_{p+n}(z) \tag{5.2}$$

where $\theta_{p+n}\geq 0$ and $\sum_{n=0}^{\infty}\theta_{p+n}=1$.

Proof. Let $f(z)$ be expressible in the form

$$f(z)=\sum_{n=0}^{\infty}\theta_{p+n}f_{p+n}(z)$$

Then

$$f(z)=z^p-\sum_{n=1}^{\infty}\frac{p-\alpha}{p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)}\theta_{p+n}z^{p+n}$$

Now

$$\sum_{n=1}^{\infty}\frac{[p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{p-\alpha}\left\{\frac{p-\alpha}{p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)}\theta_{p+n}\right\}$$

$$=\sum_{n=1}^{\infty}\theta_{p+n}=1-\theta_p\leq 1$$

Therefore, $f(z)\in k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$.

Conversely, suppose that $f(z)\in k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$. Thus

$$a_{p+n}\leq\frac{p-\alpha}{p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)},\quad (p,\ n\in\mathbf{N})$$

Setting

$$\theta_{p+n}=\frac{p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)}{p-\alpha}a_{p+n}$$

and

$$\theta_p=1-\sum_{n=1}^{\infty}\theta_{p+n}$$

we see that $f(z)$ can be expressed in the form (5.2). □

Corollary 5.2. *The extreme points of the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ are*

$$f_p(z)=z^p$$

and

$$f_{p+n}(z)=z^p-\frac{p-\alpha}{p(1+k)\delta_n(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_n(\beta,\gamma,\xi,p)}z^{p+n},\quad (p,\ n\in\mathbf{N})$$

6 Closure Theorems

Let the function $f(z)\in T(p)$ defined by (2.2), and the function

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n},\qquad (b_{p+n}\geq 0,\ p\in\mathbf{N}) \tag{6.1}$$

be in the class $T(p)$, then the class $T(p)$ is said to be convex if

$$\rho f(z) + (1-\rho)g(z) \in T(p) \tag{6.2}$$

where $0 \leq \rho \leq 1$.

Now we prove the convexity of the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$

Theorem 6.1. *The class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ is convex.*

Proof. Let $f(z)$ defined by (2.2) and $g(z)$ defined by (6.1) be in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ then

$$\rho f(z) + (1-\rho)g(z) = z^p - \sum_{n=1}^{\infty}[\rho a_{p+n} + (1-\rho)b_{p+n}]z^{p+n}$$

Applying Theorem 3.2 for the functions $f(z)$ and $g(z)$, we get

$$\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)][\rho a_{p+n} + (1-\rho)b_{p+n}]$$

$$\leq \rho(p-\alpha) + (1-\rho)(p-\rho) = (p-\alpha)$$

This completes the proof of the Theorem 6.1. □

Let the functions $f_i(z)$ defined for $i = 1, 2, \ldots, m$ by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{i,p+n}z^{p+n}, \quad (a_{i,p+n} \geq 0,\ p \in \mathbf{N}) \tag{6.3}$$

for $z \in \mathcal{U}$.

Now we prove the following theorem of functions in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha_i)$

Theorem 6.2. *Let the functions $f_i(z)$ defined by (6.3) be in the class$k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha_i)$ for each $i = 1, 2, \ldots, m$. Then the function $h(z)$ defined by*

$$h(z) = z^p - \frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m} a_{i,p+n}\right)z^{p+n} \tag{6.4}$$

is in the class $k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$ where $\alpha = \min_{1\leq i\leq m}\{\alpha_i\}$ with $0 \leq \alpha_i < p$.

Proof. Since $f_i(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha_i),\ i = 1, 2, \ldots, m$. By applying Theorem 3.2, we observe that

$$\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha_i)\delta_n(\beta,\gamma,\xi,p)]a_{i,p+n} \leq p-\alpha$$

Hence

$$\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha_i)\delta_n(\beta,\gamma,\xi,p)]\left(\frac{1}{m}\sum_{i=1}^{m} a_{i,p+n}\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=1}^{\infty}[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha_i)\delta_n(\beta,\gamma,\xi,p)]a_{i,p+n}\right)$$

$$\leq \frac{1}{m}\sum_{i=1}^{m}(p-\alpha_i) \leq p-\alpha$$

which in view of Theorem 3.2, again implies that $h(z) \in k - TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$. □

7 Inclusion Properties

In this section we give inclusion theorem for functions in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$.

Theorem 7.1. *Let $f(z)$ defined by (2.2) and $g(z)$ defined by (6.1) be in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, then the function $h(z)$ defined by*

$$h(z)=z^{p}-\sum_{n=1}^{\infty}(a_{p+n}^{2}+b_{p+n}^{2})z^{p+n} \tag{7.1}$$

is in the class $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\theta)$ where

$$\theta=p-\frac{2p(1+k)(p-\alpha)^{2}[\delta_{1}(\lambda,\mu,\eta,p)-\delta_{1}(\beta,\gamma,\xi,p)]}{[p(1+k)\delta_{1}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{1}(\beta,\gamma,\xi,p)]^{2}-2\delta_{1}(\beta,\gamma,\xi,p)(p-\alpha)^{2}} \tag{7.2}$$

with $\delta_1(\lambda,\mu,\eta,p)$ and $\delta_1(\beta,\gamma,\xi,p)$ are given by (4.3) and (4.4) respectively.

Proof. in view of Theorem 3.2, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{[p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\theta)\delta_{n}(\beta,\gamma,\xi,p)]}{p-\theta}(a_{p+n}^{2}+b_{p+n}^{2})\leq 1 \tag{7.3}$$

Since $f(z)$ and $g(z)$ belong to $k-TUCV_{\beta,\gamma,\xi}^{\lambda,\mu,\eta}(p,\alpha)$, so

$$\sum_{n=1}^{\infty}\left(\frac{p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)}{p-\alpha}\right)^{2}a_{p+n}^{2}\leq$$

$$\left(\sum_{n=1}^{\infty}\frac{p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)}{p-\alpha}a_{p+n}\right)^{2}\leq 1 \tag{7.4}$$

Also,

$$\sum_{n=1}^{\infty}\left(\frac{p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)}{p-\alpha}\right)^{2}b_{p+n}^{2}\leq$$

$$\left(\sum_{n=1}^{\infty}\frac{p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)}{p-\alpha}b_{p+n}\right)^{2}\leq 1 \tag{7.5}$$

Adding (7.4) and (7.5), we get

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)}{p-\alpha}\right)^{2}(a_{p+n}^{2}+b_{p+n}^{2})\leq 1 \tag{7.6}$$

Thus (7.3) will hold if

$$\frac{p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\theta)\delta_{n}(\beta,\gamma,\xi,p)}{p-\theta}\leq$$

$$\frac{1}{2}\left(\frac{p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)}{p-\alpha}\right)^{2}$$

That is, if

$$\theta\leq p-\frac{2p(1+k)(p-\alpha)^{2}[\delta_{n}(\lambda,\mu,\eta,p)-\delta_{n}(\beta,\gamma,\xi,p)]}{[p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)]^{2}-2\delta_{n}(\beta,\gamma,\xi,p)(p-\alpha)^{2}} \tag{7.7}$$

Since

$$A(n)=p-\frac{2p(1+k)(p-\alpha)^{2}[\delta_{n}(\lambda,\mu,\eta,p)-\delta_{n}(\beta,\gamma,\xi,p)]}{[p(1+k)\delta_{n}(\lambda,\mu,\eta,p)-(pk+\alpha)\delta_{n}(\beta,\gamma,\xi,p)]^{2}-2\delta_{n}(\beta,\gamma,\xi,p)(p-\alpha)^{2}} \tag{7.8}$$

is an increasing function of $n\in\mathbf{N}$. Letting $n=1$ in (7.8), we arrive at (7.2). □

8 Radii of K-Uniform Starlikeness, Convexity and Close-to-Convexity

For some σ $(0 \leq \sigma < p)$ and all $z \in \mathcal{U}$: A function $f(z) \in T(p)$ is said to be p-valently starlike of order σ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma \tag{8.1}$$

A function $f(z) \in T(p)$ is said to be p-valently convex of order σ if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \sigma \tag{8.2}$$

A function $f(z) \in T(p)$ is said to be p-valently close-to-convex of order σ if it satisfies

$$\mathrm{Re}\left\{f'(z)\right\} > \sigma \tag{8.3}$$

Theorem 8.1. *Let the function* $f(z)$ *defined by* (2.2) *be in the class* $k - TUCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$. *Then* $f(z)$ *is p-valently starlike of order* σ $(0 \leq \sigma < p)$ *in the disk* $|z| < r_1$ *where*

$$r_1 = \inf_{n\in\mathbf{N}}\left\{\frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p-\alpha)(n+p-\sigma)}\right\}^{1/n} \tag{8.4}$$

The result is sharp with the extremal function given by (3.5).

Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \sigma \quad (|z| < r_1) \tag{8.5}$$

Indeed we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - p\right| &= \left|\frac{-\sum_{n=1}^{\infty} na_{p+n}z^n}{1 - \sum_{n=1}^{\infty} a_{p+n}z^n}\right| \\ &\leq \frac{\sum_{n=1}^{\infty} na_{p+n}|z|^n}{1 - \sum_{n=1}^{\infty} a_{p+n}|z|^n}\end{aligned} \tag{8.6}$$

Hence (8.6) is true if

$$\sum_{n=1}^{\infty} na_{p+n}|z|^n \leq (p-\sigma) - \sum_{n=1}^{\infty}(p-\sigma)a_{p+n}|z|^n$$

That is, if

$$\sum_{n=1}^{\infty}(n+p-\sigma)a_{p+n}|z|^n \leq p - \sigma$$

or

$$\sum_{n=1}^{\infty}\left(\frac{n+p-\sigma}{p-\sigma}\right)a_{p+n}|z|^n \leq 1 \tag{8.7}$$

By Theorem 3.2, (8.6) is true if

$$\frac{n+p-\sigma}{p-\sigma}|z|^n \leq \frac{p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)}{p-\alpha} \tag{8.8}$$

Solving (8.8) for $|z|$, we get

$$|z| \leq \left\{\frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p-\alpha)(n+p-\sigma)}\right\}^{1/n}$$

or

$$r_1 = \inf_{n\in\mathbf{N}} \left\{ \frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p-\alpha)(n+p-\sigma)} \right\}^{1/n} \tag{8.9}$$

□

Similarly we can prove the following results.

Theorem 8.2. *Let the function* $f(z)$ *defined by* (2.2) *be in the class* $k-TUCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$. *Then* $f(z)$ *is p-valently convex of order* σ $(0 \le \sigma < p)$ *in the disk* $|z| < r_2$ *where*

$$r_2 = \inf_{n\in\mathbf{N}} \left\{ \frac{p(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p+n)(p-\alpha)(n+p-\sigma)} \right\}^{1/n} \tag{8.10}$$

The result is sharp with the extremal function given by (3.5).

Theorem 8.3. *Let the function* $f(z)$ *defined by* (2.2) *be in the class* $k-TUCV^{\lambda,\mu,\eta}_{\beta,\gamma,\xi}(p,\alpha)$. *Then* $f(z)$ *is p-valently close-to-convex of order* σ $(0 \le \sigma < p)$ *in the disk* $|z| < r_3$ *where*

$$r_3 = \inf_{n\in\mathbf{N}} \left\{ \frac{(p-\sigma)[p(1+k)\delta_n(\lambda,\mu,\eta,p) - (pk+\alpha)\delta_n(\beta,\gamma,\xi,p)]}{(p+n)(p-\alpha)} \right\}^{1/n} \tag{8.11}$$

The result is sharp with the extremal function given by (3.5).

Competing Interests

Authors declare that no competing interests exist.

References

Al-Kharsani, H. A., AL-Hajiry, S. S. (2006). Subordination results for the family of uniformly convex p-valent functions. J. Inequal. Pure and Appl. Math., 7(1), Art. 20.

Al-Refai, O., Darus, M. (2009). An extension to the Owa-Srivastava fractional operator with applications to parabolic starlike and uniformly convex functions. International Journal of Differential Equations, Article ID 597292, 18 pages.

Altintas, O., Irmak, H., Srivastava, H. M. (1995a). A subclass of analytic functions defined by using certain operators of fractional calculus. Comput. Math. Appl., 30, 1-9.

Altintas, O., Irmak, H., Srivastava, H. M. (1995b). Fractional calculus and certain class of starlike functions with negative coefficients. Comput. Math. Appl., 30, 9-15.

Duren, P. L. (1983). Univalent functions. Grunlehven der mathematischen wissenschaften, Bd. Vol. 259, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo.

Goodman, A. W. (1991a). On uniformly convex functions. Annal. Polo. Math. LVI. 1, 87-92.

Goodman, A. W. (1991b). On uniformly starlike functions. J. Math. Anal. Appl., 155, 364-370.

Gupta, V. P. (1984). Convex class of starlike functions. Yokohama Math. J., 32, 55-59.

Jack, I. S. (1971). Functions starlike and convex of order α. J. London Math. Soc., 23, 469-474.

Kanas, S., Wisniowska, A. (1999). Conic regions and K-uniformly convexity. J. Compu Appl. Math., 105, 327-336.

Kanas, S., Wisniowska, A. (2000). Conic regions and K-uniformly starlike functions. Per. Roumaine Math. Pures Appl., 45, 647-657.

Khairnar, S. M., Meena More. (2009). On a subclass of multivalent β- uniformly starlike and convex functions defined by a linear operator. IAENG International Journal of Applied Mathematics, 39(3), IJAM-39-06.

Ma, M., Minda, D. (1992). Uniformly convex functions. Annal. Polo. Math., LVII. 2, 165-175.

Ma, M., Minda, M. (1993). Uniformly convex functions II. Annal. Polo. Math., LVIII. 3, 275-285.

Murugusundaramoorthy, G., Themangani, R. (2009). Fractional calculus operators associated with a subclass of uniformly convex functions. Jordan J. Math. Statistics, 2(1), 1-10.

Owa, S. (1978). On the distortion theorems- I. Kyungpook. Math. J. 18, 53-59.

Owa, S. (1985). On certain classes of p-valent functions with negative coefficients. Bull. Belg. Math. Soc. Simon Stevin, 59, 385-402.

Owa, S. (1998). On uniformly convex functions. Math. Japonica, 48(3), 377-384.

Owa, S., Saigö, M., Srivastava, H. M. (1989). Some characterization involving a certain fractional integral operators. J. Math. Anal. & Appl.,140, 416-426.

Partil, D. A., Thakare, N. K. (1983). On convex hulls and extreme points of p-valent starlike and convex classes with applications. Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.), 27, 145-160.

Pinchuk, B. (1968). On starlike and convex functions of order α. Duke Math. J. 35, 721-734.

Raina, R. K., Srivastava, H. M. (1996). A certain subclass of analytic functions associated with operators of fractional calculus. Comput. Math. Appl., 32, 13-19.

Robertson, M. S. (1936). On the theory of univalent functions. Ann. Of Math. 37, 374-408.

Rønning, F. (1991). On starlike functions associated with parabolic regions. Ann. Univ. Mariae Curiesklodoweyh Sect. A45, 117-122.

Rønning, F. (1993). Uniformly convex functions a corresponding class of starlike functions. Proc. Amer. Math. Soc., 118(1), 189-196.

Schild, A. (1965). On starlike functions of order α. Amer. J. Math., 87, 65-70.

Silverman, H. (1975). Univalent functions with negative coefficients. Proc. Amer. Math. Soc., 51, 109-116.

Sokôł, J., Wisniowska, A. (2011). On certain problem in the class of k-starlike functions. Comp. & Math. with Appl., 62, 4733-4741.

Srivastava, H. M., Karlsson, P. M. (1985). Multiple Gaussian hypergeometric series. Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York/ Chichester/ Brishane/ Toronto.

Srivastava, H. M., Mishra, A. K. (2000). Applications of fractional calculus to parbolic starlike and uniformly convex functions. Comp. & Math. with Appl., 39, 57-69.

Srivastava, H. M., Owa, S. (1989). Univalent functions, fractional calcalus and their applications. Halsted Press/ John Wiley and Sons, Chichester/ New York.